



AN INTRODUCTION TO THE TOMOGRAPHIC PICTURE OF QUANTUM MECHANICS

A. Ibort^a, V.I. Man'ko^b, G. Marmo^c, A. Simoni^c, F. Ventriglia^c

^a*Departamento de Matemáticas, Universidad Carlos III de Madrid,*

Av.da de la Universidad 30, 28911 Leganés, Madrid, Spain

(e-mail: albertoi@math.uc3m.es)

^b*P.N.Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia*

(e-mail: manko@na.infn.it)

^c*Dipartimento di Scienze Fisiche dell' Università "Federico II" e Sezione INFN di Napoli,*

Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Naples, Italy

(e-mail: marmo@na.infn.it, simoni@na.infn.it, ventriglia@na.infn.it)

Abstract

Starting from the famous Pauli problem on the possibility to associate quantum states with probabilities, the formulation of quantum mechanics in which quantum states are described by fair probability distributions (tomograms, i.e. tomographic probabilities) is reviewed in a pedagogical style. The relation between the quantum state description and the classical state description is elucidated. The difference of those sets of tomograms is described by inequalities equivalent to a complete set of uncertainty relations for the quantum domain and to nonnegativity of probability density on phase space in the classical domain. Intersection of such sets is studied. The mathematical mechanism which allows to construct different kinds of tomographic probabilities like symplectic tomograms, spin tomograms, photon number tomograms, etc., is clarified and a connection with abstract Hilbert space properties is established. Superposition rule and uncertainty relations in terms of probabilities as well as quantum basic equation like quantum evolution and energy spectra equations are given in explicit form. A method to check experimentally uncertainty relations is suggested using optical tomograms. Entanglement phenomena and the connection with semigroups acting on simplexes are studied in detail for spin states in the case of two qubits. The star-product formalism is associated with the tomographic probability formulation of quantum mechanics.

Key words Quantum tomograms, entangled states.

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1 Introduction

Pure quantum states are usually associated with wave functions [1] or vectors in a Hilbert space [2]. Mixed quantum states are associated with density matrices [3] or density states [4]. Pauli [5, 6] posed the problem whether it was possible to associate quantum states with probability distributions as it happens in classical statistical mechanics. Pauli problem was more concrete, namely, is it possible to reconstruct the quantum state (i.e., the wave function) from the knowledge of the probability distribution for the position and the probability distribution for the momentum? The answer to this particular question is negative (see the discussion in Reichenbach's book [7] and a recent example in [8]). But the general idea of Pauli to associate quantum states with probability distributions was implemented by introducing the tomographic probability representation of quantum states [9]. This representation is based on the Radon transform [10] of Wigner function [11], suggested in [12, 13] to connect the measurable optical tomographic probability [14, 15] to reconstruct the Wigner function of a photon quantum state. The mathematical nature of the tomographic probability representation was clarified in [16, 17] (but see also, e.g., [18, 19, 20, 21, 22, 23]). In quantum mechanics we have the conventional Heisenberg and Schrödinger representations. The tomographic probability representation is another one. The physical properties of quantum systems can be studied in the tomographic probability representation as well as in the Heisenberg and Schrödinger representations or in the Feynman representation [24] based on the use of path integral as main ingredient of the quantum picture. Since the tomographic picture deals with probabilities which describe the quantum states, we will show how important quantum aspects as uncertainty relations and superposition principle can be described in terms of tomographic probabilities. We suggest a method of checking the Heisenberg uncertainty relations using quantum state tomograms. It is worthy to mention that all the available representations of quantum mechanics are equivalent (see, e.g., the review [25]). One cannot say that some representation is better or worse than the others. Nevertheless, each representation has peculiar properties due to which some quantum aspects become clearer and simpler than in other representations. For sure, the superposition principle can be formulated in the easiest and clearest form using the natural linear structure of a Hilbert space whose vectors are realized by complex wave functions. The tomographic probability picture is very natural for problems of quantum information and quantum entanglement, so each picture has its own merits.

We shall try to present in a pedagogical style the construction of the tomographic probability representation both for discrete (spin, qubit) variables, studied in [26, 27], and for continuous variables like position and momentum following [8, 16, 17, 28, 29, 30]. The tomographic probability can be used also in classical statistical mechanics [31, 32]. The tomogram of a classical state is the Radon transform of the standard probability density on the classical phase space. In this setting both classical and quantum states can be described by tomographic probability distributions. The difference between classical and

quantum states in the tomographic description is related to the different physical constraints the state tomograms have to satisfy in order to be either in the classical or in the quantum domain. In the tomographic representation the quantum-classical relation is formulated in terms of properties of the tomographic probability densities. It will be obvious that the ambient space of Radon transforms of functions defined on the phase space contains the subset of quantum tomograms (tomograms admissible in the quantum domain), the subset of classical tomograms (those admissible in the classical domain) and the subset of those which are not admissible neither in quantum nor in classical domain. The subsets of quantum and classical tomograms have a not empty intersection. In this work we study the properties of tomograms providing a characterization of classical and quantum domains as well as of their intersection.

The paper is organized as follows. After a preliminary section 2 in which the Pauli problem is considered and its tomographic solution is introduced, there are two main parts. Part one is devoted to discuss quantum mechanics formulations on phase space and their relations to tomography. It contains nine sections. Weyl systems are discussed in section 3.1. Wigner functions are considered in section 3.2, and their transformation properties under the action of the automorphisms of the Weyl-Heisenberg group are discussed in section 3.3. Tomograms and Radon transform are studied in section 3.4. Classical and quantum probability distributions are considered in section 3.5. State reconstruction procedure is studied in section 3.6. Tomographic families of rank-one projectors are introduced in section 3.7, while a general and abstract setting of tomographic maps is presented in section 3.8. Finally, a unified approach to construct the most commonly used tomographic families of observables is given in section 3.9. In the second part we describe quantum mechanics in the tomographic picture. It contains six sections. In the sections 4.1 and 4.2, the superposition rule and the uncertainty relations respectively are discussed in the tomographic probability representation. In the section 4.3 some examples of classical or quantum distributions are presented. In section 4.4 we study basic equations (time evolution and eigenvalue equations) in tomographic representation. An application to entanglement and separability on examples of two qubit states is given in section 4.5. Bell inequalities and association with a semigroup structure are considered in section 4.6. In section 5 some conclusions and perspectives are finally drawn.

2 The Pauli problem: the original formulation and the tomographic solution

In his book [6], after having introduced the wave function in the position representation by means of $\psi(\vec{x})$ and in momentum representation by means of $\varphi(\vec{p})$, along with the probability densities

$$W(\vec{x}) = |\psi(\vec{x})|^2 = \psi(\vec{x})^* \psi(\vec{x}); \quad W(\vec{p}) = \varphi(\vec{p})^* \varphi(\vec{p}), \quad (1)$$

Pauli claims: These functions $\psi(\vec{x})$ and $\varphi(\vec{p})$, which are usually called ‘probability amplitudes’ are not, however, directly observable with regard to their phases; this holds only for the probability densities $W(\vec{x})$ and $W(\vec{p})$. In a footnote Pauli states: The mathematical problem as to whether for given functions $W(\vec{x})$ and $W(\vec{p})$, the wave function ψ , if such a function exists, is always uniquely determined (i.e. if $W(\vec{x})$ and $W(\vec{p})$ are physically compatible) has still not being investigated in all its generality. [6]

As it was shown in Reichenbach’s book [7], the negative answer to the original Pauli problem may be given by means of counterexamples. For instance [8], consider the two squeezed states with wave functions in the position representation ($\hbar = 1$):

$$\begin{aligned}\psi_1(q) &= N \exp(-\alpha q^2 + i\beta q), \\ \psi_2(q) &= N \exp(-\alpha^* q^2 + i\beta q), \\ \text{Re } \alpha \geq 0 \quad ; \quad \beta &= \beta^* \quad ; \quad N = \sqrt[4]{\frac{\alpha + \alpha^*}{\pi}}\end{aligned}\tag{2}$$

and in the momentum representation

$$\begin{aligned}\psi_1(p) &= \frac{N}{\sqrt{2\alpha}} \exp\left[-\frac{(\beta + p)^2}{4\alpha}\right], \\ \psi_2(p) &= \frac{N}{\sqrt{2\alpha^*}} \exp\left[-\frac{(\beta + p)^2}{4\alpha^*}\right].\end{aligned}\tag{3}$$

One then has

$$\begin{aligned}|\psi_1(q)|^2 &= |\psi_2(q)|^2 = |N|^2 \exp[-(\alpha + \alpha^*)q^2], \\ |\psi_1(p)|^2 &= |\psi_2(p)|^2 = \frac{|N|^2}{2|\alpha|} \exp\left[-(\beta + p)^2 \left(\frac{\alpha + \alpha^*}{4\alpha\alpha^*}\right)\right].\end{aligned}\tag{4}$$

The fidelity $f = |\langle\psi_1|\psi_2\rangle|^2$ is

$$f = \frac{\alpha + \alpha^*}{2\sqrt{\alpha\alpha^*}}.$$

One can see that $f \neq 1$, which means that the states are different.

To understand the reasons why the knowledge of the two marginal distributions of position and momentum are not sufficient for reconstructing a state, consider the family of dimensionless observables, depending on two real parameters μ, ν :

$$X(\mu, \nu) = \mu Q + \nu P, \tag{5}$$

where, restoring the Planck constant \hbar , Q and P generate the Weyl–Heisenberg algebra $[Q, P] = i\hbar I$. The spectrum of $X(\mu, \nu)$ is the real line, which we parametrize by X , with corresponding improper eigenvector $|X\mu\nu\rangle$. In the position representation,

$$X(\mu, \nu) = -i\hbar\nu \frac{d}{dq} + \mu q, \tag{6}$$

and its (improper) eigenfunctions may be chosen as

$$\varphi_{X\mu\nu}(q) = \langle q|X\mu\nu\rangle = N e^{-i\frac{\mu}{2\hbar\nu}q^2 + i\frac{X}{\hbar\nu}q}, \quad N = \frac{1}{\sqrt{2\pi\hbar|\nu|}} \quad (7)$$

The normalization is such that $\langle X'|X\rangle = \delta(X - X')$. Analogously, in the momentum representation, the eigenfunctions may be chosen as

$$\tilde{\varphi}_{X\mu\nu}(p) = \langle p|X\mu\nu\rangle = \frac{1}{\sqrt{2\pi\hbar|\mu|}} e^{i\frac{\nu}{2\hbar\mu}p^2 - i\frac{X}{\hbar\mu}p} \quad (8)$$

Now, define the symplectic tomogram of a (normalized) pure state $|\psi\rangle$ with respect to the family $X(\mu, \nu)$ as (see, e.g., [33])

$$\begin{aligned} \mathcal{T}_\psi(X, \mu, \nu) &= |\langle X\mu\nu|\psi\rangle|^2 = \frac{1}{2\pi\hbar|\nu|} \left| \int \psi(q) e^{i\frac{\mu}{2\hbar\nu}q^2 - i\frac{X}{\hbar\nu}q} dq \right|^2, \quad \nu \neq 0 \\ &= \frac{1}{2\pi\hbar|\mu|} \left| \int \tilde{\psi}(p) e^{-i\frac{\nu}{2\hbar\mu}p^2 + i\frac{X}{\hbar\mu}p} dp \right|^2, \quad \mu \neq 0 \end{aligned} \quad (9)$$

where the Fourier transform of the wave function has been introduced

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(y) e^{-i\frac{py}{\hbar}} dy. \quad (10)$$

In other words, $\mathcal{T}_\psi(X, \mu, \nu)dX$ is the marginal probability such that a measure in the given state $|\psi\rangle$ of the observable $X(\mu, \nu)$, with fixed μ, ν , has values in $(X, X + dX)$. Of course, as $|\psi\rangle$ is normalized,

$$\int \mathcal{T}_\psi(X, \mu, \nu)dX = 1 \quad (11)$$

independently of the chosen values of μ, ν .

Then, it is apparent that the Pauli problem amounts to reconstruct the given state $|\psi\rangle$ from the knowledge of the two marginal probability distributions of position, $\mathcal{T}_\psi(X, 1, 0)$, and momentum, $\mathcal{T}_\psi(X, 0, 1)$. As a matter of fact, the following reconstruction formula holds (see, e.g., [34]) for the density matrix $\rho(q, q') = \psi(q)\psi^*(q')$:

$$\begin{aligned} \rho(q, q') &= \frac{1}{2\pi} \int \mathcal{T}_\psi(X, \mu, \frac{q - q'}{\hbar}) e^{i[X - \frac{\mu}{2}(q + q')]} dX d\mu \\ &= \frac{1}{(2\pi)^2} \int \mathcal{T}_\psi(X, \mu, \nu) e^{\frac{i}{\hbar}p(q - q')} \exp[i(X - \mu q - \nu p)] dX d\mu d\nu dp \end{aligned} \quad (12)$$

Thus, the answer to the Pauli problem is negative because the reconstruction requires the knowledge of many different marginal probability distributions, corresponding to many different observables of the family $X(\mu, \nu)$. A minimal set of such observables is called a *quorum* [35], and the characterization of a tomographic quorum will be discussed in the next sections.

3 Quantum mechanics on phase space and tomography

3.1 Weyl systems

The notion of Weyl system is useful as a tool to formulate a quantum version of classical hamiltonian mechanics by using the symplectic form on a linear classical phase space. We will use it in the following as a suitable setting to discuss the quantum and classical tomographic maps and their relations. Here we briefly recall the definition of Weyl system, mainly to fix notations. Hereafter $\hbar = 1$.

Given a symplectic vector space (V, ω) , where V has even dimension $2n$ and ω is a nondegenerate skew-symmetric bilinear form on it, a Weyl system is a strongly continuous map from V to the group of unitary operators on some Hilbert space \mathcal{H} :

$$W : V \rightarrow \mathcal{U}(\mathcal{H}) \quad (13)$$

satisfying the condition

$$W(v_1)W(v_2)W^\dagger(v_1 + v_2) = Ie^{\frac{i}{2}\omega(e_1, e_2)}. \quad (14)$$

It is a projective unitary representation of the Abelian vector group associated with V .

A theorem due to von Neumann [36] establishes that such a map exists for any finite dimensional symplectic vector space. Indeed, the Hilbert space \mathcal{H} can be realized as the space of square integrable functions on any Lagrangian subspace of V . By using a Lagrangian subspace L , this is a subspace L of dimension half of the dimension of the space V and such that $\omega(x, y) = 0$ for all $x, y \in L$, and its dual L^* it is possible to decompose V into $V \cong L \oplus L^* = T^*L$. Because L is Lagrangian, Eq. (14) implies that the unitary operators corresponding to these vectors will commute. We consider the Lebesgue measure on L and we construct the following specific realization of W . The action of the restrictions $W|_L, W|_{L^*}$ of the map W to the subspaces L and L^* , on $\mathcal{L}^2(L, d^n x)$ is given by:

$$(W|_L(y)\psi)(x) = \psi(x + y), \quad (W|_{L^*}(\alpha)\psi)(x) = e^{i\alpha(x)}\psi(x), \quad (15)$$

for $x, y \in L$, $\alpha \in L^*$, $\psi \in \mathcal{L}^2(L, d^n x)$.

The strong continuity requirement in the definition of W allows to use Stone theorem to get

$$W(tv) = e^{itR(v)}, \quad \forall v \in V, \quad (16)$$

with $R(v)$ the infinitesimal generator of the one parameter unitary group $W(tv)$, $t \in \mathbb{R}$, depending linearly on v .

In the following we shall be mainly concerned with the simple case $V = \mathbb{R}^2$, where, for simplicity, we introduce (q, p) coordinates. Then, by denoting

$$P = R(1, 0); \quad Q = R(0, 1) \quad (17)$$

we may also represent the Weyl map in the familiar form $v = (q, p) \rightarrow W(v) = D(q, p)$, where the displacement operator $D(q, p)$ is given by the formula:

$$D(q, p) = \exp [i (pQ - qP)] , \quad (18)$$

which turns out to be an irreducible representation of the Weyl–Heisenberg group (see section 3.3 for more details on this topic).

By introducing complex coordinates $z = \frac{1}{\sqrt{2}}(q + ip)$ for V , we may write the displacement operator, eq. (18), in terms of creation and annihilation operators as:

$$D(z) = \exp (za^\dagger - z^*a) . \quad (19)$$

Generalizations of previous formulae to the case $V = \mathbb{R}^{2n}$ are obvious, just by considering n –dimensional vector operators.

Thus, a Weyl system provides, for any choice of a Lagrangian subspace, canonical pairs of operators and the displacements operators, i. e. a projective irreducible representation of the translation group. It is now possible to associate an operator with any function f on V admitting a (symplectic) Fourier transform \tilde{f} . Consider

$$f(q, p) = \int d^n \alpha d^n x \tilde{f}(\alpha, x) \exp[i(\alpha q - xp)] . \quad (20)$$

The operator $W(f)$ associated to f is obtained by a clever use of the Fourier transform, substituting the (symplectic) Fourier kernel $\exp[i(\alpha q - xp)]$ with the Weyl displacement operator $D(\alpha, x) = \exp [i (\alpha Q - xP)]$:

$$W(f) = \int d^n \alpha d^n x \tilde{f}(\alpha, x) \exp [i (\alpha Q - xP)] . \quad (21)$$

The above formula defines a unitary isomorphism between the space of the Hilbert-Schmidt operators on $\mathcal{L}^2(L, d^n x)$ and the square integrable functions of $\mathcal{L}^2(L \oplus L^*, d^n x d^n \alpha)$.

3.2 Wigner functions

We have considered a Weyl system to be a projective unitary representation of an Abelian vector group V of even dimension. Another useful interpretation of a Weyl system comes from the following considerations.

Consider a fiducial vector $|\psi_0\rangle$ in the Hilbert space \mathcal{H} carrying the projective unitary representation of V . We may consider an immersion of V into \mathcal{H} by means of the map

$$V \ni v \mapsto |v\rangle = W(v) |\psi_0\rangle . \quad (22)$$

We denote this map by $\mathcal{W}_0 : V \rightarrow \mathcal{H}$. The image of \mathcal{W}_0 is a submanifold of \mathcal{H} , it is not a subspace. If we consider a fiducial operator $A_0 \in \text{End}(\mathcal{H})$, we may in a similar way immerse V into the space of operators acting on \mathcal{H} by setting

$$\mathcal{W}_0 : V \rightarrow \text{End}(\mathcal{H}) , \quad v \mapsto W^\dagger(v) A_0 W(v) =: A(v) .$$

As it happens with any immersion of a manifold \mathcal{M} into a manifold \mathcal{N} , we can consider the pull-back to V of the covariant tensor fields on \mathcal{H} . As a matter of fact, on account of the probabilistic interpretation of quantum mechanics, the immersion of V should be considered to take place into the manifold of rays, i.e. $\mathcal{R}(\mathcal{H})$ the manifold of rays in \mathcal{H} or the complex projective space associated with \mathcal{H} .

With any immersion $\phi : \mathcal{M} \rightarrow \mathcal{N}$ we have a map $\phi^* : \mathcal{F}(\mathcal{N}) \rightarrow \mathcal{F}(\mathcal{M})$ which, however, need not be surjective. Moreover, dealing with infinite dimensional manifolds (\mathcal{H} or $\text{End}(\mathcal{H})$) the properties of the pulled-back tensors depend on the specific immersion we deal with. In particular we may require the map $v \mapsto |v\rangle$, or $v \mapsto A(v)$, to satisfy appropriate measurability, continuity or differentiability properties. Depending on the use of the pulled-back tensors we may prefer one immersion over the other and, for instance, prefer the Schrödinger picture over the Heisenberg picture.

By using the decomposition of V into different Lagrangian subspaces, say $L \times L'$ (we use the cartesian product notation to stress we are considering it as a manifold rather than as a vector space), we may consider ‘eigenvectors’ of the position operators, say $|q\rangle$ or $|q'\rangle$. In this way we may ‘pull-back’ any vector $|\psi\rangle$ to a wave function $\psi(q) = \langle q|\psi\rangle$ or $\psi(q') = \langle q'|\psi\rangle$; similarly for an observable A we have the ‘matrix coefficient’ $\langle q|A|q'\rangle = f_A(q, q')$.

Had we chosen the decomposition $V = L \times L^*$, i.e. using ‘eigenvectors’ $|q\rangle$ and $|p\rangle$, we would have had $\langle q|A|p\rangle$ or $\langle p|A|q\rangle$, classically this corresponds to the use of boundary values, q and q' , or initial Cauchy data, q and p . These various matrix coefficient functions are connected by means of the completeness relations

$$\int |q\rangle dq \langle q| = I = \int |p\rangle dp \langle p|. \quad (23)$$

This general idea of considering the representation as an immersion of a manifold V into the Hilbert space \mathcal{H} [37], allows to consider the pull-back of the algebra structure on the operators. We may define a star-product by setting

$$\phi^*(A) * \phi^*(B) := \phi^*(AB). \quad (24)$$

In this way, on the subspace of functions we obtain by means of the pull-back of the operators a nonlocal and noncommutative product.

In the Schrödinger picture we would have

$$f_A(v) = \frac{\langle v|Av\rangle}{\langle v|v\rangle}, \quad (25)$$

we could also consider

$$\tilde{f}_A(v) = \langle \psi_0|W^\dagger(v)A_0W(v)|\psi_0\rangle, \quad (26)$$

and so on. In general, these functions are called symbols of the corresponding operators and carry a specific qualification to keep track of the specific immersion one is considering (we have Weyl symbols, Berezin symbols, Hörmander symbols, and so on).

At this point it should be clear that if we deal also with unbounded operators on \mathcal{H} we end up with a variety of situations already at the level of topologies we are willing to consider on $\text{End}(\mathcal{H})$. Additional problems will arise from the specific choice of the fiducial vector we start with (if we need to take derivatives, we better deal with smooth or analytic vectors [38]). Usually, it is better to leave some of these choices unsettled and take them into full account only in each specific problem.

Having clarified some aspects connected with the pull-back of observables, let us turn now to the pull-back of states. We recall that states are usually considered to be normalized positive functionals on the space of observables. States are not a vector space but we may consider convex combinations. Pure states are those that cannot be written as convex combination. To avoid some pathologies, very often states are also required to be normal.

Thanks to Gleason's theorem, states are also called density operators, however this may be misleading because it may give the impression that they should be considered with the same topologies as the operator algebra. However the star-product we have considered allows to distinguish the \mathcal{L}^1 -algebra, associated with operators, which acts on the \mathcal{L}^2 -space of states. Thus, while the pull-back of states or observables always provides us with functions on V , the subsets to which they belong enjoy quite different properties and therefore it is advisable to avoid considering them as mathematical entities of the same kind.

The distinction will reappear to be crucial when we would like to compare with the analogous situation in classical mechanics described on the same phase space V . Again here states and observables are quite different, pure classical states end up being distributions, while observables, due to the possibility of taking Poisson brackets, are usually required to be smooth functions. This distinction plays a very relevant role when we consider their associated Radon transforms to represent tomographic states or tomographic observables.

We may now define a Wigner function the way it was considered by Wigner. Given a state $|\psi\rangle$ we form the rank-one projector

$$\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \quad (27)$$

which defines a normalized positive functional on the space of observables. We may consider, as we have stressed, either the density state

$$\rho(q, q') = \frac{\langle q|\psi\rangle\langle\psi|q'\rangle}{\langle\psi|\psi\rangle} \quad (28)$$

or the function

$$\tilde{\mathcal{W}}(q, p) := \frac{\langle q|\psi\rangle\langle\psi|p\rangle}{\langle\psi|\psi\rangle}, \quad (29)$$

i.e. the matrix coefficients of the rank-one projector in the (position, position) representation or in the (position, momentum) representation. By using the completeness relation we have

$$\langle\psi|q'\rangle = \int \langle\psi|p\rangle dp \langle p|q'\rangle \quad (30)$$

which provides us with the transformation from one representation to another.

It is now feasible to use convex combinations of pure states to define generic states and their associated Wigner representations. The Wigner function $\mathcal{W}(p, q)$ of a density state $\rho(q, q')$ is defined, restoring the Planck constant \hbar , as:

$$\mathcal{W}(q, p) := \int \rho(q + \frac{x}{2}, q - \frac{x}{2}) \exp(-\frac{i}{\hbar} p x) dx, \quad (31)$$

and it results

$$\mathcal{W}(q, p) = 2 \exp(\frac{i}{\hbar} 2pq) \int \tilde{\mathcal{W}}(q', p') \exp\left[\frac{i}{\hbar} (q' p' - 2pq' - 2qp')\right] \frac{dq' dp'}{\sqrt{2\pi\hbar}}. \quad (32)$$

Now a simple manipulation, changing the variable $x/2 = s$ and extracting from the integral the bra $\langle\psi|$ and ket $|\psi\rangle$, we get:

$$\mathcal{W}(q, p) = 2 \langle\psi| \left[\int e^{-2ips/\hbar} |q - s\rangle \langle q + s| ds \right] |\psi\rangle. \quad (33)$$

3.3 Transformation properties of Wigner functions

We would like to understand the transformation properties of Wigner functions under linear symplectic maps and dilations on phase space. The exploration of these issues will suggest the possibility of extending the definition of Wigner's function to the space of irreducible representations of the Weyl–Heisenberg group and to discern their homogeneity dependence on Planck's constant \hbar . We will make these comments precise in what follows.

The Weyl map allows to associate automorphisms ν_ϕ on the space of unitary operators with elements ϕ of the symplectic linear group $Sp(V, \omega)$ of V , according to the following diagram

$$\begin{array}{ccc} V & \xrightarrow{W} & \mathcal{U}(\mathcal{H}) \\ \phi \downarrow & & \downarrow \nu_\phi \\ V & \xrightarrow{W} & \mathcal{U}(\mathcal{H}) \end{array} \quad (34)$$

for any $\phi \in Sp(V, \omega)$, by setting

$$\nu_\phi(W(v)) = W(\phi(v)) = U_\phi^\dagger W(v) U_\phi, \quad \forall v \in V. \quad (35)$$

Recall that an isomorphism $\phi: V \rightarrow V$ is symplectic if $\omega(\phi(u), \phi(v)) = \omega(u, v)$ for all $u, v \in V$. Moreover, the group of all symplectic isomorphisms of V can be identified with the matrix symplectic group $Sp(n)$ by choosing a symplectic basis on V . In other words, the automorphism ν_ϕ , corresponding to the symplectic linear transformation ϕ of V , is a inner automorphism of the group of unitary operators, that is there exists a unitary operator U_ϕ such that $\nu_\phi(V) = U_\phi^\dagger V U_\phi$ for all $V \in \mathcal{U}(\mathcal{H})$, because it belongs to the connected component of the identity of the automorphism group [39]. At the level of the infinitesimal generators of the unitary group, we have

$$U_\phi^\dagger R(v) U_\phi = R(\phi(v)). \quad (36)$$

Further insight on the physical meaning of the Wigner function of a density state ρ , was obtained from its representation as the expectation value of the shifted parity operator $\mathcal{P}(q, p)$ (see, e.g., Royer [40]). In fact, we can write the expression given by Eq. (33) of the Wigner function $\mathcal{W}(q, p)$ corresponding to the state $|\psi\rangle$, as:

$$\mathcal{W}(q, p) = 2 \langle \psi | \mathcal{P}(q, p) | \psi \rangle, \quad (37)$$

with $\mathcal{P}(q, p)$ being the shifted parity operator:

$$\mathcal{P}(q, p) = \int e^{-2ips/\hbar} |q - s\rangle \langle q + s| ds. \quad (38)$$

Notice that $\mathcal{P}(0, 0) = \int |-s\rangle \langle s| ds$ is just the parity operator defined as $(\mathcal{P}\psi)(q) = \psi(-q)$ or, equivalently the unitary operator \mathcal{P} satisfying:

$$\mathcal{P}Q\mathcal{P} = -Q, \quad \mathcal{P}P\mathcal{P} = -P. \quad (39)$$

Now, we get immediately that:

$$\mathcal{P}(q, p) = D(q, p) \mathcal{P} D(q, p)^\dagger, \quad (40)$$

where the displacement operators $D(q, p)$ have the usual form given by Eq. (18). Then, the Wigner function corresponding to the pure state $|\psi\rangle$ can be readily written in the form:

$$\mathcal{W}(q, p) = 2 \langle \psi | D(q, p) \mathcal{P} D(q, p)^\dagger | \psi \rangle, \quad (41)$$

and for a given density state ρ we obtain:

$$\mathcal{W}_\rho(p, q) = 2 \text{Tr} [\rho D(p, q) \mathcal{P} D^\dagger(p, q)] = 2 \text{Tr} [\rho D(2p, 2q) \mathcal{P}]. \quad (42)$$

Because the displacement operators provide a specific irreducible representation of the Weyl–Heisenberg group, the previous formula makes apparent the possibility of generalizing Wigner’s function as a function on the space of irreducible representations of the Weyl–Heisenberg group. In fact the Weyl–Heisenberg group $\text{WH}(n)$, for $n = 1$, may be presented as the group of triples of real numbers (p, q, t) with the composition law:

$$(q, p, t) \circ (q', p', t') = (q + q', p + p', t + t' + \frac{1}{2}(pq' - qp')). \quad (43)$$

The associated canonical operators with their commutation relations $[Q, P] = iI$ are a realization of the Lie algebra of the Weyl–Heisenberg group, and a irreducible unitary representation is provided by:

$$U(q, p, t) = D(q, p) e^{itI}. \quad (44)$$

In the general case $n \geq 1$, the irreducible representations of $\text{WH}(n)$ are parametrized up to a unitary equivalence by a real parameter γ [41]. Kirillov’s theory of coadjoint orbits [42] provides a natural way to construct them. In fact,

Kirillov's theorem establishes that for nilpotent groups there is a one-to-one correspondence between coadjoint orbits of the group and equivalence classes of unitary irreducible representations of it. It is easy to check that for the Weyl–Heisenberg group the space of coadjoint orbits has two strata, the regular one whose coadjoint orbits are copies of the symplectic linear space (V, ω) and are labelled by $\gamma \neq 0$, and the singular stratum, corresponding to the label $\gamma = 0$ whose coadjoint orbits are points, hence giving rise to trivial representations. The parameter γ weights the central element of the group and it can be easily read out from a given irreducible representation looking at $U_\gamma(0, t) = e^{i\gamma t}$ and therefore the action of $\text{Aut}(\text{WH}(n))$ on the set of irreducible representations can be analyzed.

In order to introduce a generalized notion of Wigner functions [43] for representations with $\gamma \neq 1$, we have to choose previously a representative U_γ out of any equivalence class $[U]_\gamma$. We choose, for $n = 1$, the representatives for $\gamma > 0$ as:

$$U_\gamma(q, p, t) := U_{\gamma=1}(\sqrt{\gamma}q, \sqrt{\gamma}p, \gamma t) = D(\sqrt{\gamma}q, \sqrt{\gamma}p)e^{i\gamma t}. \quad (45)$$

Once a representation U_γ has been chosen, the parity operator \mathcal{P} given by Eq. (39) may be expressed as:

$$\mathcal{P} = \frac{\gamma}{2} \int \frac{dqdp}{2\pi} D(\sqrt{\gamma}q, \sqrt{\gamma}p) = \frac{1}{2} \int \frac{dqdp}{2\pi} D(q, p). \quad (46)$$

From this expression the properties:

$$\mathcal{P} = \mathcal{P}^\dagger; \quad \mathcal{P}U_\gamma(q, p, t)\mathcal{P} = U_\gamma(-q, -p, t), \quad (47)$$

readily follow.

Now, given γ , we define the associated (generalized) Wigner function of a density state ρ as

$$\begin{aligned} \mathcal{W}_\rho(q, p; \gamma) &: = 2 \text{Tr} [\rho U_\gamma(q, p, t) \mathcal{P} U_\gamma^\dagger(q, p, t)] \\ &= 2 \text{Tr} [\rho D(2\sqrt{\gamma}q, 2\sqrt{\gamma}p) \mathcal{P}] \\ &= \mathcal{W}_\rho(\sqrt{\gamma}q, \sqrt{\gamma}p; 1). \end{aligned} \quad (48)$$

We remark that, while the dependence on the parameter t disappears and the function is invariant on the subgroup $(0, t)$, a new dependence on the representation label γ appears.

We now consider the action of a dilation $\phi_\lambda : \phi_\lambda(q, p, t) = (\lambda q, \lambda p, \lambda^2 t)$. Then, as a result of our choice of the representatives U_γ , we get

$$U_\gamma(\phi_\lambda(q, p, t)) = U_\gamma(\lambda q, \lambda p, \lambda^2 t) = U_{\lambda^2 \gamma}(q, p, t). \quad (49)$$

So, the Wigner function transforms as:

$$\begin{aligned} \mathcal{W}_\rho(\phi_\lambda(q, p); \gamma) &= \mathcal{W}_\rho(\lambda q, \lambda p; \gamma) \\ &= 2 \text{Tr} [\rho U_{\lambda^2 \gamma}(q, p, t) \mathcal{P} U_{\lambda^2 \gamma}(q, p, t)] \\ &= \mathcal{W}_\rho(q, p; \lambda^2 \gamma), \end{aligned} \quad (50)$$

while

$$\int \frac{\lambda^2 \gamma dq dp}{2\pi} \mathcal{W}_\rho(\lambda q, \lambda p; \gamma) = \int \frac{\lambda^2 \gamma dq dp}{2\pi} \mathcal{W}_\rho(q, p; \lambda^2 \gamma) = \text{Tr } \rho. \quad (51)$$

The dilation transformation may be more interestingly written as:

$$\mathcal{W}_\rho \left(\lambda q, \lambda p; \frac{\gamma}{\lambda^2} \right) = \mathcal{W}_\rho(q, p; \gamma). \quad (52)$$

We observe that the dilation $(\lambda q, \lambda p, \lambda^2 t)$ yields the expected dilation γ/λ^2 on the label γ , which is ‘dual’ of the parameter t . For an infinitesimal dilation $\lambda = 1 + \epsilon$ we may expand:

$$\begin{aligned} \mathcal{W}_\rho(q, p; \gamma) &= \mathcal{W}_\rho \left((1 + \epsilon)(q, p); \frac{\gamma}{(1 + \epsilon)^2} \right) \\ &= \mathcal{W}_\rho(q, p; \gamma) + \epsilon \left[\mathbf{v} \frac{\partial \mathcal{W}_\rho}{\partial \mathbf{v}}(q, p; \gamma) - 2\gamma \frac{\partial \mathcal{W}_\rho}{\partial \gamma}(q, p; \gamma) \right] + O(\epsilon^2), \end{aligned} \quad (53)$$

where we have used the notation $\mathbf{v} = (q, p)$ and

$$\mathbf{v} \frac{\partial \mathcal{W}_\rho}{\partial \mathbf{v}} = q \frac{\partial \mathcal{W}_\rho}{\partial q} + p \frac{\partial \mathcal{W}_\rho}{\partial p}. \quad (54)$$

Then we obtain the following differential equation for the Wigner function:

$$\mathbf{v} \frac{\partial \mathcal{W}_\rho}{\partial \mathbf{v}}(\mathbf{v}; \gamma) - 2\gamma \frac{\partial \mathcal{W}_\rho}{\partial \gamma}(\mathbf{v}; \gamma) = 0. \quad (55)$$

So far, we have put $\hbar = 1$. It is possible however to study the dependence on \hbar by using the displacement operators given, instead of Eq. (18), by the expressions:

$$D_\hbar(q, p) = \exp \left[i \left(\frac{p}{\sqrt{\hbar}} \frac{Q}{\sqrt{\hbar}} - \frac{q}{\sqrt{\hbar}} \frac{P}{\sqrt{\hbar}} \right) \right], \quad (56)$$

and the canonical commutation relations:

$$\frac{1}{\hbar} [Q, P] = iI, \quad (57)$$

while t gives place to t/\hbar and the unitary representation given by Eq. (44) becomes:

$$U \left(\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}, \frac{t}{\hbar} \right) = D_\hbar(q, p) e^{i \frac{t}{\hbar} I} = \exp \left(i \frac{R(q, p)}{\hbar} \right) e^{i \frac{t}{\hbar} I}, \quad (58)$$

so that eventually we get the above formulae with γ replaced by γ/\hbar everywhere. In particular, for the Wigner function we have:

$$\mathcal{W}_\rho(\sqrt{\gamma} \mathbf{v}; 1) = \mathcal{W}_\rho(\mathbf{v}; \gamma) \longrightarrow \mathcal{W}_\rho \left(\sqrt{\frac{\gamma}{\hbar}} \mathbf{v}; 1 \right) = \mathcal{W}_\rho \left(\mathbf{v}; \frac{\gamma}{\hbar} \right). \quad (59)$$

Under the action of a dilation,

$$(\lambda \mathbf{v}, \lambda^2 t) \rightarrow (\lambda \mathbf{v}; \gamma/\lambda^2) \rightarrow (\lambda \mathbf{v}; \gamma/\lambda^2 \hbar) \quad (60)$$

and we may choose $\gamma = 1$, to get a differential equation for the Wigner function $\mathcal{W}_\rho(\mathbf{v}; \frac{1}{\hbar})$ corresponding to the infinitesimal ‘dilation’ $(\lambda \mathbf{v}; 1/\lambda^2 \hbar)$.

Notice that the scaling properties Eq.s (59, 60) is consistent with the dependence on \hbar of the Wigner function. We recall that the density state ρ has the dimension of an inverse length ℓ^{-1} , where $\ell \sim \sqrt{\hbar}/\sqrt{m\omega}$: so $\rho(x, x') = \ell^{-1} \rho'(x/\ell, x'/\ell)$. Then it is easy to check that the following property holds:

$$\mathcal{W}\left(\frac{q}{\lambda}, \frac{p}{\lambda}, \frac{1}{\lambda^2 \hbar}\right) = \mathcal{W}\left(q, p, \frac{1}{\hbar}\right). \quad (61)$$

We refer to [43] for more details and further results on this direction.

3.4 Tomograms, Wigner functions and Radon Transform

By means of the reconstruction formula (12) the Wigner function $\mathcal{W}(p, q)$ of a density state $\rho(q, q')$ may be recast in the form:

$$\mathcal{W}(p, q) = \frac{\hbar}{2\pi} \int \mathcal{T}(X, \mu, \nu) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu. \quad (62)$$

The above equation explicitly contains the Planck constant \hbar , to be coherent with Eq. (12). Hereafter $\hbar = 1$, however. We recall that in general the Wigner function is not a fair probability distribution as it is not non-negative; nevertheless, it is a function on the phase space of the system, and its reconstruction formula (62) is just the Radon anti-transform of the tomogram.

The Radon transform [10] originally was formulated to solve the problem of reconstructing a function $f(p, q)$ from its integrals on arbitrary straight lines $\mu q + \nu p = X$ in the (q, p) -plane

$$\int f(p, q) \delta(X - \mu q - \nu p) dp dq =: (\mathcal{R}f)(X, \mu, \nu). \quad (63)$$

Here δ is the Dirac delta function and the parameters X, μ, ν are real. The homogeneity property follows from the properties of the delta function. The inverse transform reads:

$$f(p, q) = \frac{1}{(2\pi)^2} \int (\mathcal{R}f)(X, \mu, \nu) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu. \quad (64)$$

Remark: Additional hypotheses, such as global integrability conditions, are required to guarantee the uniqueness of the inverse transform [44]. We point out that here a subclass of functions is selected by requiring that our ‘manipulations’ provide us with an injective map.

In a general sense, we may call the Radon transform (63), the tomogram of the function $f(p, q)$, therefore the ambient space for tomographic states is

provided by the range of the Radon transform when it is applied to properly chosen functions on phase space.

The problem we address now is the following. Let us select two classes of functions on the phase space satisfying special conditions. The first class of functions consists of all probability distribution densities on phase space (the $q - p$ plane) describing states of classical particles. The second class consists of all the Wigner functions describing quantum states thought of as rank-one projectors. We study tomogram properties of these two classes. There exists also a class of tomograms which are Radon transforms of the Weyl symbol of observables. These tomograms are not probability densities.

So, the symplectic tomogram we dealt with in section 2 may be eventually interpreted as the Radon transform of the Wigner function

$$\begin{aligned}\mathcal{T}(X, \mu, \nu) &= \int \frac{1}{2\pi} \mathcal{W}(p, q) \delta(X - \mu q - \nu p) dp dq \\ &= \int \rho(y, y') \varphi_{X\mu\nu}^*(y) \varphi_{X\mu\nu}(y') dy dy' .\end{aligned}\tag{65}$$

The standard description of classical states with fluctuations is given by a non-negative joint probability distribution function $f(p, q)$ on the phase space (a plane, for a particle with one degree of freedom). The function is normalized, i.e.

$$\int f(p, q) dp dq = 1 .\tag{66}$$

The classical state tomogram $(\mathcal{R}f)(X, \mu, \nu)$ can be written in the form

$$(\mathcal{R}f)(X, \mu, \nu) = \langle \delta(X - \mu q - \nu p) \rangle_f\tag{67}$$

where the average is done using the probability distribution $f(p, q)$ in the phase space [31, 32]. The tomogram is the probability distribution function in a rotated and scaled reference frame on the phase space. It can be expressed in terms of a scaling parameter s and a rotation parameter θ :

$$\mu = s \cos \theta , \quad \nu = s^{-1} \sin \theta .\tag{68}$$

For fixed μ and ν one then gets a line $X = \mu q + \nu p$ in the plane (q, p) with an orientation θ from the position axis. Thus the physical meaning of the variable X is that it is the ‘position’ of the particle measured in the reference frame of the phase-space whose axes are rotated by an angle θ with respect to the old reference frame, after preliminary canonical scaling of the initial position $q \rightarrow sq$ and momentum $p \rightarrow s^{-1}p$. The coordinates X and $Y = -s^2\nu q + s^{-2}\mu p$ provide a canonical transformation preserving the symplectic form in the phase space. For that reason the classical tomogram is called ‘symplectic’.

In the quantum case, Eq. (65) can be written in a form similar to Eq. (67):

$$\mathcal{T}_\rho(X, \mu, \nu) = \langle \delta(X - \mu Q - \nu P) \rangle_\rho .\tag{69}$$

The difference with Eq. (67) is that here the position and momentum are quantum operators Q and P , and therefore we have to take into account uncertainty relations. The averaging in Eq. (69) is done using a density state ρ , i.e.

$$\langle A \rangle_\rho := \text{Tr}(\rho A). \quad (70)$$

For fixed μ and ν , the operator

$$X(\mu, \nu) = \mu Q + \nu P \quad (71)$$

together with its conjugate

$$Y(\mu, \nu) = -s^2 \nu Q + s^{-2} \mu P \quad (72)$$

satisfies the canonical commutation relations of the Weyl–Heisenberg algebra: $[X(\mu, \nu), Y(\mu, \nu)] = [Q, P]$. The observable $X(\mu, \nu)$ is a new position operator, i.e. the position after a symplectic (linear canonical) transformation in the quantum non-commutative phase space (Q, P) of the particle. The real variable X gives the possible results of a measure of $X(\mu, \nu)$ and runs over the spectrum of $X(\mu, \nu)$. In this way a description of quantum tomograms is recovered in complete analogy with the classical case. So the tomogram is also ‘symplectic’ in the quantum case, it is associated with an automorphism of the Weyl–Heisenberg algebra.

In classical mechanics, the transition from the distribution function of two canonically conjugate variables (position q and momentum p) to the distribution function of a (μ, ν) –family of position variables (position X) does not play a crucial role, due to absence of quantum mechanical constraints like the uncertainty relations of Heisenberg [45] and Schrödinger–Robertson [46, 47, 48, 49]. On the contrary, the use of tomograms in quantum mechanics should encode the properties required to allow for the uncertainty relations.

So, the tomograms of all admissible functions $f(p, q)$ form an ambient space (here admissibility means only that the Radon transform exists and is one-to-one). This space contains the subset of probability densities. In turn, the subset of the probability densities contains two subsets. One subset contains the Radon transforms of Wigner functions which are probability densities (quantum domain). The other one contains the Radon transforms of classical probability distributions on phase space. These two subsets have a not empty intersection. Both these subsets are embedded into the total set of tomograms, which therefore contains tomographic functions corresponding neither to classical nor to quantum states.

3.5 Distributions and quasi-distributions: classical and quantum

As we have argued, on the same space an object like a symplectic tomogram $\mathcal{T}(X, \mu, \nu)$ may determine a state both in classical and in quantum domain. Let us discuss some difference which exist for these two domains in the context of the tomographic description. State tomograms in both domains must satisfy the following common requirements:

1. Nonnegativity: $\mathcal{T}(X, \mu, \nu) \geq 0$.
2. Integrability: $\int \mathcal{T}(X, \mu, \nu) dX < \infty$.
3. Homogeneity: $\mathcal{T}(\lambda X, \lambda\mu, \lambda\nu) = \frac{1}{|\lambda|} \mathcal{T}(X, \mu, \nu)$.

Other properties of the symplectic state tomograms are required to distinguish the states in quantum and classical domains. For example, the necessary condition for the tomogram of a classical state is the nonnegativity of its Radon anti-transform, i. e.

$$\int \mathcal{T}(X, \mu, \nu) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu \geq 0. \quad (73)$$

The violation of this inequality means that the tomogram does not describe a classical state. On the other side the condition for a symplectic tomogram to describe a quantum state can be formally written in an analogous way, as

$$\int \mathcal{T}(X, \mu, \nu) \exp[i(X - \mu Q - \nu P)] dX d\mu d\nu \geq 0, \quad (74)$$

which means that the operator obtained from the above Radon anti-transform, being a density state, must be a non-negative normalized functional on observables. If the inequality is violated, the tomogram does not describe a quantum state. Tomograms satisfying both conditions belong to the intersection of quantum and classical domain, so they may be chosen as starting Cauchy datum of either quantum or classical time evolutions. For example, the gaussian tomogram of coherent or squeezed and correlated states [49, 50, 51] satisfies both inequalities.

Now, the question arises if the previous requirements of nonnegativity, integrability and homogeneity are sufficient to select only classical or quantum tomograms. In other words, there exist tomograms satisfying the three requirements, but violating both inequalities? It would mean that these tomograms need not describe a state, neither a classical one nor a quantum one if additional requirements are not met. An example of such a kind of tomograms may be manufactured to answer in the affirmative this question. The example is provided by scaling the parameters μ, ν of the tomogram of the first excited state of an harmonic oscillator

$$\mathcal{T}_1(X, \mu, \nu) = \frac{2e^{-\frac{1}{\mu^2 + \nu^2} X^2}}{\sqrt{\pi(\mu^2 + \nu^2)}} \frac{X^2}{\mu^2 + \nu^2} \quad (75)$$

by means of a real parameter λ obtaining

$$\mathcal{T}_\lambda(X, \mu, \nu) = \frac{2e^{-\frac{1}{(\lambda\mu)^2 + (\lambda\nu)^2} X^2}}{\sqrt{\pi((\lambda\mu)^2 + (\lambda\nu)^2)}} \frac{X^2}{(\lambda\mu)^2 + (\lambda\nu)^2}. \quad (76)$$

This new tomogram is still positive, integrable and homogeneous. But the quantum inequality is not fulfilled by \mathcal{T}_λ for $\lambda \neq 1$, as discussed in [52] where, in a

different contest, it is shown that the fidelity of the scaled tomogram and the genuine tomogram of the harmonic oscillator ground state is negative: $f = -2|\lambda|^2$ for small λ . Also, the classical inequality is violated by the scaled tomogram, because its Radon anti-transform yields a distribution (generalized function) which is negative for small q, p .

In conclusion, the set of non-negative, integrable and homogeneous tomographic functions is divided into three parts: one containing tomograms of quantum states, another one containing those of classical states and a third part containing neither. The first two parts intersect each other in the domain of tomograms satisfying both quantum and classical inequalities. The third part does not intersect the others, and contains tomographic functions describing neither quantum nor classical states therefore they violate both inequalities. As a consequence, more constraints than the previous ones are needed to unambiguously select quantum, or classical, tomograms in order to give a tomographic version of quantum mechanics fully equivalent to the usual formulations and yielding the classical mechanics in an appropriate limit. At least in principle, for the symplectic case they are sufficiently described by the quantum, or classical, inequality.

As intrinsic characterization of classical tomogram is the property that the Fourier components of the classical tomograms is non-negative (see Eq. (73)). The property of quantum tomograms given by inequality (74) is also intrinsic but needs the additional construction of operators Q and P on a Hilbert space. One can formulate the inequality in a form equivalent to Eq. (74) without introducing these operators. This is accomplished replacing the nonnegativity condition of the operator with Sylvester criterion for the matrix of the operator corresponding to the integral kernel of Eq. (74). It means that in any basis the principal minors of such a matrix are non-negative. This condition is given in form of algebraic integral inequalities, not containing formally mention of a Hilbert space.

Another intrinsic description of the set of quantum tomograms can be formulated in the following way. The tomograms of coherent states are gaussian and belong to the intersection of quantum and classical sets. The intersection is invariant under the operation of taking convex sums of the tomograms. Due to the completeness property of coherent states, the composition of all coherent state tomograms given by Eq.s (137), (138) provides the set of all quantum tomograms corresponding to pure quantum states. Then, combining with convex sums these pure quantum states, we get any mixed state tomogram. In this way the whole set of quantum tomograms is recovered.

3.6 Tomographic sets and state reconstruction

From the conceptual point of view, Pauli's problem raises a central question in the formulation of quantum mechanics. In general, any system may be described by a set \mathcal{O} of observables and a 'dual' set of density states \mathcal{S} , which together give rise to probability measures on the real axis, they are just the probability distributions of the values of the observables in the states.

In the usual formulation of quantum mechanics, any observable A , i.e., an hermitian operator, uniquely determines a projector valued measure (PVM) $P_A(E)$ on the sets E of the Borel σ -algebra of the real line [53, 54], so that from a density state ρ a probability measure $m_{\rho,A}$ can be defined as

$$m_{\rho,A}(E) := \text{Tr}(\rho P_A(E)) . \quad (77)$$

It is just the probability that the mean value of the observable A in the state ρ belongs to E . As a consequence:

$$m_{\rho,A}(\mathbb{R}) = 1 . \quad (78)$$

The mean value of A on the state ρ , when it exists, can be written as an integral over a real variable λ with respect to that probability measure:

$$\text{Tr}(\rho A) = \int \lambda m_{\rho,A} . \quad (79)$$

This for pure states $\rho = |\psi\rangle\langle\psi|$ reads

$$\langle\psi|A|\psi\rangle = \int \lambda m_{\rho,A} . \quad (80)$$

and a functional calculus for hermitian operators can be constructed by defining the operator $f(A)$ as:

$$\langle\psi|f(A)|\psi\rangle = \int f(\lambda) m_{\rho,A} . \quad (81)$$

for any integrable function f . So, the knowledge of $m_{\rho,A}$ for any state ρ and fixed A allows for the reconstruction of $\langle\psi|A|\psi\rangle$ for all ψ in the domain of A and therefore of A . In fact, the matrix elements of A in any chosen basis of its domain are given by the polarization identity:

$$\langle\psi|\varphi\rangle = \frac{1}{4} \left[\|\psi + \varphi\|^2 + \|\psi - \varphi\|^2 - i \|\psi + i\varphi\|^2 + i \|\psi - i\varphi\|^2 \right] . \quad (82)$$

Viceversa, when ρ is fixed, the knowledge of $m_{\rho,A}$ for any observable A , in particular for all projectors $|\psi\rangle\langle\psi|$, allows for the reconstruction of $\text{Tr}(\rho|\psi\rangle\langle\psi|) = \langle\psi|\rho|\psi\rangle$ and therefore, by polarization, of ρ .

For instance, when A is the position Q , the associated projectors $P_Q(E)$ act on wave functions as a multiplication by the characteristic function of the Borel set E :

$$(P_Q(E)\psi)(x) = \chi_E(x)\psi(x) \quad (83)$$

Now, a density state ρ can be spectrally decomposed in terms of rank-one projectors as (because selfadjoint compact operator):

$$\rho = \sum_k \alpha_k P_k ; \quad \alpha_k \geq 0, \quad \sum_k \alpha_k = 1, \quad (84)$$

therefore the previous formulae became:

$$\begin{aligned}
m_{\rho,Q}(E) &:= \text{Tr}(\rho P_Q(E)) = \sum_k \alpha_k \text{Tr}(P_k P_Q(E)) \\
&= \sum_k \alpha_k \int \chi_E(x) |\psi_k(x)|^2 dx = \sum_k \alpha_k \int_E |\psi_k(x)|^2 dx \\
&= \int_E \sum_k \alpha_k |\psi_k(x)|^2 dx = \int_E \rho(x, x) dx.
\end{aligned} \tag{85}$$

This shows that the probability measure $m_{\rho,Q}(E)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} , with Radon-Nykodim derivative

$$\rho(x, x) = \sum_k \alpha_k |\psi_k(x)|^2 \tag{86}$$

which is just the diagonal part of the density matrix in the position representation. Then the mean value of the position operator is:

$$\text{Tr}(\rho Q) = \int x \rho(x, x) dx \tag{87}$$

or, for pure states $|\psi\rangle$:

$$\langle \psi | Q | \psi \rangle = \int x |\psi(x)|^2 dx. \tag{88}$$

Thus, Pauli's problem may be reformulated as: To determine the state from the knowledge of a pair of probability measures $m_{\rho,A}$, i.e. when A is the position Q or the momentum P . This set is not sufficient, while the symplectic tomography provides a set of observables $X(\mu, \nu) = \mu Q + \nu P$ which is sufficient for the reconstruction. Notably, the symplectic set is generated by the position operator Q under the action of a family of unitary transformations $S(\mu, \nu)$. Introducing the auxiliary parameters λ, θ as

$$\mu = e^\lambda \cos \theta, \nu = e^{-\lambda} \sin \theta, \tag{89}$$

we can write

$$S(\mu, \nu) = \exp \left[\frac{i\lambda}{2} (QP + PQ) \right] \exp \left[\frac{i\theta}{2} (Q^2 + P^2) \right], \tag{90}$$

so that

$$S(\mu, \nu) Q S^\dagger(\mu, \nu) = \mu Q + \nu P. \tag{91}$$

Thus, the transformation $S(\mu, \nu)$ yields the appropriate probability measure associated to the observable $X(\mu, \nu)$

$$m_{\rho, X(\mu, \nu)}(E) := \text{Tr}(\rho S(\mu, \nu) P_Q(E) S^\dagger(\mu, \nu)) = \int_E \langle X | S^\dagger(\mu, \nu) \rho S(\mu, \nu) | X \rangle dX \tag{92}$$

whose density is just the tomographic probability distribution

$$\mathcal{T}_\rho(X, \mu, \nu) := \langle X | S^\dagger(\mu, \nu) \rho S(\mu, \nu) | X \rangle = \text{Tr}(\rho S(\mu, \nu) | X \rangle \langle X | S^\dagger(\mu, \nu)) \tag{93}$$

where the kets $|X\rangle$ are the eigenkets of $Q : Q |X\rangle = X |X\rangle$.

3.7 Rank-one projectors as tomographic sets

In view of formula (93) we may consider in general tomographic, i.e. (possibly over-) complete, sets of rank-one projectors. In a sense, they are the elementary ‘building blocks’ of any tomography. Moreover, as we will show in the following, tomographic family of rank-one projectors allow to clarify readily the ingredients of a tomographic reconstruction formula.

We start with an abstract finite dimensional case. Assume the Hilbert space of the vector states \mathcal{H} to be n -dimensional, so that rank-one projectors span a n^2 -dimensional Hilbert space $\mathbb{H} = \mathcal{H} \otimes \mathcal{H}^*$, containing all the density states as well as the (bounded) operators on \mathcal{H} , i.e. $\mathbb{H} = B(\mathcal{H})$. The scalar product is given by the trace: $\langle A|B \rangle = \text{Tr}(A^\dagger B)$. A *minimal* tomographic set is a basis $\{P_k\}, k \in \{1, \dots, n^2\}$, of rank-one projectors, which may be orthonormalized by a Gram-Schmidt procedure:

$$|V_j\rangle = \sum_{k=1}^{n^2} \gamma_{jk} |P_k\rangle \quad , \quad \langle V_i|V_j\rangle = \delta_{ij} \quad .$$

In general, every $|V_j\rangle$ is a linear combination of projectors, rather than a single projector like $|P_k\rangle$. Then a resolution of the super-unity on \mathbb{H} in terms of the P ’s reads as

$$\begin{aligned} \hat{\mathbb{I}}_{n^2} &= \sum_{i=1}^{n^2} |V_i\rangle \langle V_i| = \sum_{i,j,l=1}^{n^2} \gamma_{il}^* \gamma_{ij} P_j \text{Tr}(P_l \bullet) \\ &= \sum_{l=1}^{n^2} |G_l\rangle \langle P_l| = \sum_{j=1}^{n^2} |P_j\rangle \langle G_j| \end{aligned} \quad (94)$$

where the dual set of Gram-Schmidt operators $\{G_k\}, k \in \{1, \dots, n^2\}$, has been introduced:

$$|G_l\rangle = \sum_{i=1}^{n^2} \gamma_{il}^* |V_i\rangle = \sum_{i,j=1}^{n^2} \gamma_{il}^* \gamma_{ij} |P_j\rangle. \quad (95)$$

We observe that G_l is a nonlinear function of the projectors $\{P_k\}$, because also the coefficients γ ’s depend on the projectors. Moreover:

$$\begin{aligned} \langle P_i|G_l\rangle &= \sum_{j=1}^{n^2} \gamma_{jl}^* \langle P_i|V_j\rangle = \sum_{j,k=1}^{n^2} \gamma_{jl}^* (\gamma^*)_{ik}^{-1} \langle V_k|V_j\rangle \\ &= \sum_{j=1}^{n^2} \gamma_{jl}^* (\gamma^*)_{ij}^{-1} = \delta_{il}. \end{aligned} \quad (96)$$

Thus, Eq. (94) shows that a resolution of the (super-) identity, associated with a tomographic reconstruction formula, is determined by a pair of dual sets, the P ’s and the G ’s. Besides the role of the dual operators may be interchanged

in such a formula, so that a tomography would be better defined in terms of a pair of dual sets. In the context of the harmonic analysis and of the wavelet signal analysis, these dual sets are known as dual frames [55].

Formulae similar to the previous ones hold for any tomographic, i.e. (over-) complete, set of rank-one projectors. In fact, along with the minimal tomographic set discussed above, that is a quorum of rank-one projectors, it is very useful to deal with over-complete or even maximal sets of rank-one projectors, obtained for instance by acting on a fiducial projector P_0 with a unitary representation of a Lie group. The previous summations became then integrations over the orbit Ω through P_0 .

A suitable illustration of such tomographic sets is given by the qubit (spin 1/2) tomography over a 2-dimensional Hilbert space of the vector states \mathcal{H} . Out of the standard basis vectors $\{|m\rangle\}$, where $m = \pm 1/2$ is the spin projection on the z -axis, a fiducial projector $P_0 = |m\rangle\langle m|$ is rotated by means of the operators U of an irreducible representation of the group $SU(2)$: the tomogram of any operator ρ is defined as

$$\mathcal{T}_\rho(m, U) = \text{Tr}(U|m\rangle\langle m|U^\dagger \rho) = \langle m|U^\dagger \rho U|m\rangle .$$

Then Ω is the orbit of the co-adjoint group action in the dual of the algebra, that is the Bloch sphere S^2 of all rank-one projectors. The projector $U|m\rangle\langle m|U^\dagger$, with U parametrized by the usual Euler angles (θ, ϕ, ψ) , corresponds to a point on S^2 determined by a unit vector $\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. In other words, we can use the (θ, ϕ) parametrization to write a generic projector in matrix form as ($m = 1/2$)

$$P(\theta, \phi) = \frac{1}{2} \begin{bmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 - \cos \theta \end{bmatrix} . \quad (97)$$

Then, the corresponding resolution of the identity reads [16]:

$$\hat{\mathbb{I}} = \int_0^{2\pi} \int_0^\pi |G(\theta, \phi)\rangle \langle P(\theta, \phi)| \sin \theta d\theta d\phi , \quad (98)$$

where in matrix form:

$$G(\theta, \phi) = \frac{1}{4\pi} \begin{bmatrix} 1 + 3 \cos \theta & 3e^{-i\phi} \sin \theta \\ 3e^{i\phi} \sin \theta & 1 - 3 \cos \theta \end{bmatrix} , \quad (99)$$

so that, for any density state ρ we get the dual reconstruction formulae:

$$\rho = \int_0^{2\pi} \int_0^\pi G(\theta, \phi) \text{Tr}(P(\theta, \phi) \rho) \sin \theta d\theta d\phi \quad (100)$$

$$= \int_0^{2\pi} \int_0^\pi P(\theta, \phi) \text{Tr}(\hat{G}(\theta, \phi) \rho) \sin \theta d\theta d\phi , \quad (101)$$

We remark that, however, in the infinite dimensional case the relation $\mathbb{H} = B(\mathcal{H})$ is no more valid and there are several relevant spaces. In particular

the Hilbert space at our disposal is the space of the Hilbert-Schmidt operators $\mathbb{H} \subset B(\mathcal{H})$, which is the typical setting of the frame theory [56], while the finest tomographic sets of rank-one projectors have to be complete both in such a Hilbert space and in the Banach space of the trace class operators. We will not insist here on the topological subtleties of the infinite dimensional case, they are discussed, e.g., in [17, 57].

3.8 General aspects of tomography

The Pauli's problem and a possible solution have been previously analyzed within the machinery of spectral analysis of selfadjoint operators. In this section, inspired by that analysis, we provide a general setting for tomography together with some general considerations.

When \mathcal{S} is the set of states of a physical system and \mathcal{A} a suitable subset of the observables \mathcal{O} , a tomography \mathcal{T} is a map from $\mathcal{S} \times (\mathcal{A} \subset \mathcal{O})$ into the set of probability measures on the real line \mathbb{R} . It is required that \mathcal{T} is such that if the probability measures $\mathcal{T}(\rho, A)$ are known for all $A \in \mathcal{A}$ it is possible, at least in principle, to reconstruct ρ .

If $\mathcal{A} = \mathcal{O}$ a tomography is available, via spectral analysis, as shown previously. In this case \mathcal{A} is a huge linear space of selfadjoint operators, bounded or not.

However, as we have seen, a tomography is available even if \mathcal{A} is restricted to a subset $\mathcal{O}' \subset \mathcal{O}$ of all the rank-one projectors. Now \mathcal{O}' is a unit spherical surface in the Hilbert space, in general infinite dimensional. It may be more useful to restrict \mathcal{A} to a subset of \mathcal{O} specified by some (multi-) parameter μ which varies in some index set \mathcal{M} . \mathcal{A} must still be such that the reconstruction of any state ρ is possible. The tomogram $\mathcal{T}(\rho, A)$ appears now as $\mathcal{T}(\rho, \mu)(E) = \mathcal{T}(\rho, A_\mu)(E) = \text{Tr}(\rho P_{A_\mu}(E))$ and is a probability measure: $0 \leq \mathcal{T}(\rho, \mu)(E) \leq 1$, for any Borel set E of reals.

In principle the elements in \mathcal{A} can be selfadjoint operators with different spectra but it is more convenient to deal with iso-spectral operators, so that a spectrum $\sigma \subset \mathbb{R}$ is associated with \mathcal{A} . In this case, for any ρ, μ we have that $\mathcal{T}(\rho, \mu)(E) = 0$ for all sets E which do not intersect the spectrum σ and $\mathcal{T}(\rho, \mu)(E) = 1$ for sets E containing the spectrum σ .

If σ is a purely continuous spectrum, as in the symplectic tomography, and $\mathcal{T}(\rho, \mu)$ is absolutely continuous with respect to the Lebesgue measure, $\mathcal{T}(\rho, \mu)$ will appear as $\mathcal{T}(X, \rho, \mu)dX$, where $X \in \sigma$ is the spectral variable, with the property $\int_\sigma \mathcal{T}(X, \rho, \mu)dX = 1$. In other terms, $\mathcal{T}(X, \rho, \mu)$ is a probability density function on the real line concentrated on the spectrum σ .

If σ is a pure point spectrum, $\mathcal{T}(\rho, \mu)$ will be concentrated on the points X_k of σ , with $\mathcal{T}(\rho, \mu)(X_k) = \text{Tr}(\rho P_{X_k, \mu}) = \mathcal{T}(k, \rho, \mu)$ a positive function of k such that $\sum_k \mathcal{T}(k, \rho, \mu) = 1$. Now $\{\mathcal{T}(k, \rho, \mu)\}_k$ can be regarded as a probability vector $\vec{\mathcal{T}}$ with a finite or countable number of components, i.e. a vector with non-negative components whose sum is one. When the spectrum is non-degenerate finite, the number n of components of $\vec{\mathcal{T}}$ is just the dimension of the Hilbert space. Geometrically, these n components may be considered as the coordinates $\{\tau_k\}$

of a point belonging to the simplex $\tau_1 + \tau_2 + \dots + \tau_n = 1$ of \mathbb{R}^n . If the spectrum is degenerate, the number of components of $\vec{\mathcal{T}}$ is less than the dimension of the Hilbert space. For instance, if \mathcal{A} is composed by rank-one projectors, $\sigma = \{0, 1\}$ and $\vec{\mathcal{T}}$ has two components. However, we notice that the qubit tomographic set presented in the previous subsection is a non-degenerate case: given the component $\tau_1 = \mathcal{T}_\rho(m, u)$, the other one is $\tau_2 = 1 - \mathcal{T}_\rho(m, u)$ and the probability vector belongs to the simplex $\tau_1 + \tau_2 = 1$.

3.9 Weyl systems and tomography

As we have seen, tomographic families of iso-spectral observables can be obtained by conjugation of a fiducial observable A_0 by a parameterized set of unitary operators U_μ , which may be the elements of a unitary (possibly square integrable [22]) representation of some Lie group \mathcal{G} and μ will then be a point on a finite dimensional manifold. In this section the most common tomographic schemes are obtained in an unified approach, by using a particular representation of a Lie group \mathcal{G} .

We discussed in section 3.3 how the Weyl map allows to associate inner automorphisms on the space of unitary operators with symplectic isomorphisms of the symplectic linear space (V, ω) . Moreover if \mathcal{G} is a Lie group and $T: \mathcal{G} \rightarrow Sp(V, \omega)$ is a linear representation of the elements of the group by symplectic maps, then we can associate an inner automorphism ν_{T_g} of the group of unitary operators to any element g of the Lie group by using the analog of Eq. (35):

$$\nu_{T_g}(W(v)) = W(T_g v) = U_g^\dagger W(v) U_g . \quad (102)$$

where now U_g denotes the unitary operator associated to T_g realizing the inner automorphism ν_{T_g} . In other words the automorphism ν_{T_g} , corresponding to the symplectic linear transformation T_g of V , is a inner automorphism of the unitary operators, as it belongs to the component of the automorphism group which is connected with the identity [39]. At the level of the infinitesimal generators of the unitary group, we have

$$U_g^\dagger R(v) U_g = R(T_g v) . \quad (103)$$

The equation above allows to make contact with most used tomographic schemes. For instance, by acting with a rotation of an angle θ followed by a squeezing e^λ we have

$$T_g = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} e^\lambda \cos \theta & e^\lambda \sin \theta \\ -e^{-\lambda} \sin \theta & e^{-\lambda} \cos \theta \end{bmatrix} \quad (104)$$

Then, by

$$U_g R(v) U_g^\dagger = R(T_g^\dagger v) . \quad (105)$$

and

$$T_g^\dagger = \begin{bmatrix} e^\lambda \cos \theta & -e^{-\lambda} \sin \theta \\ e^\lambda \sin \theta & e^{-\lambda} \cos \theta \end{bmatrix} \quad (106)$$

we obtain [58, 59]:

$$\begin{bmatrix} U_g Q U_g^\dagger \\ U_g P U_g^\dagger \end{bmatrix} = \begin{bmatrix} e^\lambda \cos \theta & e^{-\lambda} \sin \theta \\ -e^\lambda \sin \theta & e^{-\lambda} \cos \theta \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} e^\lambda \cos \theta Q + e^{-\lambda} \sin \theta P \\ -e^\lambda \sin \theta Q + e^{-\lambda} \cos \theta P \end{bmatrix}. \quad (107)$$

In this way, bearing in mind Eq.s (89÷91), we recover the symplectic tomographic family of operators $X(\mu, \nu)$:

$$U_g Q U_g^\dagger = e^\lambda \cos \theta Q + e^{-\lambda} \sin \theta P = \mu Q + \nu P = S(\mu, \nu) Q S^\dagger(\mu, \nu) = X(\mu, \nu). \quad (108)$$

We get a general, unified approach by using the inhomogeneous symplectic linear group, which is a semi-direct product of the translation group and the linear symplectic group. In that case we have only projective representations, whose generators are $I, Q, P, \frac{1}{2}(P^2 + Q^2), \frac{1}{2}(PQ + QP), \frac{1}{2}(P^2 - Q^2)$. Then most common tomographic schemes may be obtained by acting with the group on its generators as fiducial operators A_0 .

For instance, by acting on Q , $\sigma = \mathbb{R}$, we get

$$\begin{aligned} D(q, p) \tilde{S}(\alpha, \mu, \nu) Q \tilde{S}^\dagger(\alpha, \mu, \nu) D^\dagger(q, p) \\ = D(q, p) (\xi Q + \eta P) D^\dagger(q, p) = (\xi Q + \eta P) + (\xi q - \eta p) I \end{aligned} \quad (109)$$

with

$$\xi = \mu \cosh \alpha + \nu \sinh \alpha; \quad \eta = \nu \cosh \alpha + \mu \sinh \alpha, \quad (110)$$

and where, restoring the explicit dependence of μ, ν on the parameters λ, θ , the transformation $\tilde{S}(\alpha, \mu, \nu) \leftrightarrow \tilde{S}(\alpha, \lambda, \theta)$ reads:

$$\tilde{S}(\alpha, \lambda, \theta) = \exp \left[i \frac{\alpha}{2} (P^2 - Q^2) \right] \exp \left[i \frac{\lambda}{2} (PQ + QP) \right] \exp \left[i \frac{\theta}{2} (P^2 + Q^2) \right].$$

Remark: Equivalently we could have acted on P , $\sigma = \mathbb{R}$, obtaining again the same tomographic family:

$$\begin{aligned} D(q, p) \tilde{S}(\alpha, -\lambda, \theta - \frac{\pi}{2}) P \tilde{S}^\dagger(\alpha, -\lambda, \theta - \frac{\pi}{2}) D^\dagger(q, p) \\ = D(q, p) (\xi Q + \eta P) D^\dagger(q, p) = (\xi Q + \eta P) + (\xi q - \eta p) I. \end{aligned} \quad (111)$$

We may recover from the above tomographic family of observables the usual symplectic tomographic scheme [18, 60]. We do that by quotienting the whole inhomogeneous group with respect to its subgroup generated by the displacement operators times the hyperbolic rotations, and choosing a section

$$q = 0, p = 0, \alpha = 0. \quad (112)$$

Besides, by acting on the number operator

$$\frac{1}{2}(P^2 + Q^2) - \frac{1}{2} = a^\dagger a = N, \quad \sigma = \mathbb{N}_0, \quad (113)$$

we get the photon number tomography [61, 62, 63] on a section

$$\alpha = 0, \lambda = 0, \theta = 0, \quad (114)$$

while on the other section

$$q = 0, p = 0, \alpha = 0, \quad (115)$$

the so called squeeze tomography [64] is recovered.

In the coherent state tomography the fiducial operator is the projector on the vacuum state: $a|0\rangle = 0$,

$$A_0 = |0\rangle\langle 0|, \sigma = \{0, 1\} \quad (116)$$

and the tomographic family of operators is generated by the displacement operators depending on a complex parameter z , $\mathcal{D}(z) = \exp[za^\dagger - z^*a]$, eq (19), or equivalently, two real parameters. Thus one obtains a tomogram which is formally equivalent to the Husimi function, while one of the dual reconstruction formulae is the Sudarshan's diagonal coherent state representation of an operator [57].

We can also obtain spin tomography. For this, we observe that unitary irreducible representations of $SU(2)$ of dimension $m \geq 2$ are a subgroup of $SU(m)$, which in turn is isomorphic to $SO(2m) \cap Sp(2m, \mathbb{R})$, the intersection of $2m$ -dimensional rotations and linear symplectic transformations. Now, for definiteness, assume $V = \mathbb{R}^4$ with coordinates (x_1, x_2, p_1, p_2) in the Weyl diagram, with T a 2-dimensional unitary irreducible representations of $SU(2)$, then for any $g \in SU(2)$:

$$\begin{array}{ccc} V & \xrightarrow{W} & \mathcal{U}(\mathcal{H}) \\ T_g \downarrow & & \downarrow \nu_{T_g} \\ V & \xrightarrow{W} & \mathcal{U}(\mathcal{H}) \end{array} \quad (117)$$

The matrices T_g are real, orthogonal symplectic 4×4 matrices while the unitary operators U_g corresponding to the inner automorphism ν_{T_g} are generated by the hermitian operators

$$\begin{aligned} J_1 &= \frac{1}{2}(P_1Q_2 - P_2Q_1) \\ J_2 &= \frac{1}{2}(P_1P_2 + Q_1Q_2) \\ J_3 &= \frac{1}{2}\left[\frac{1}{2}(P_1^2 + Q_1^2) - \frac{1}{2}(P_2^2 + Q_2^2)\right] \\ &= \frac{1}{2}[H(P_1, Q_1) - H(P_2, Q_2)] \end{aligned} \quad (118)$$

which satisfy the commutation relations $[J_h, J_k] = i\epsilon_{hkl}J_l$. The operators $Q_a, P_a, a = 1, 2$, generate the displacement operator of the Weyl system.

In fact, choosing the lagrangian manifold to be $\{(x_1, x_2)\}$, or $\{(\rho, \varphi)\}$ in polar coordinates, then J_1 acts as $\frac{1}{2}(i\partial/\partial\varphi)$ on $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$ and its eigenfunctions are

$\psi(\rho)e^{in\varphi}$ so that its spectrum is the set of the integer and semi-integer numbers. J_3 is a semi-difference of two isotropic harmonic oscillators, so it has the same spectrum $\frac{1}{2}\{n_1 - n_2\}$. Finally, by a rotation of coordinates:

$$\begin{aligned}\xi &= \frac{1}{\sqrt{2}}(x_1 + x_2) \quad ; \quad \eta = \frac{1}{\sqrt{2}}(x_1 - x_2), \\ x_1 &= \frac{1}{\sqrt{2}}(\xi + \eta) \quad ; \quad x_2 = \frac{1}{\sqrt{2}}(\xi - \eta), \\ \frac{\partial}{\partial x_{1,2}} &= \frac{\partial \xi}{\partial x_{1,2}} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x_{1,2}} \frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \xi} \pm \frac{\partial}{\partial \eta} \right),\end{aligned}\tag{119}$$

also J_2 becomes a semi-difference of two isotropic harmonic oscillators:

$$J_2 = \frac{1}{2} \left[\frac{1}{2} \left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right) + \frac{1}{2} (\xi^2 - \eta^2) \right] = \frac{1}{2} [H(P_\xi, \xi) - H(P_\eta, \eta)] \tag{120}$$

and obviously the spectrum is the same.

To analyze the decomposition of the infinite dimensional representation of $SU(2)$ in irreducible ones we evaluate the Casimir J^2 . We start with

$$J_1^2 = \frac{1}{4} \{P_1^2 Q_2^2 + P_2^2 Q_1^2 - [P_1 Q_2 P_2 Q_1 + P_2 Q_1 P_1 Q_2]\} \tag{121}$$

and from $[Q_a, P_b] = i\delta_{ab}$, $a, b = 1, 2$, we have:

$$\begin{aligned}J_1^2 &= \frac{1}{4} \{P_1^2 Q_2^2 + P_2^2 Q_1^2 - [P_1 (P_2 Q_2 + i) Q_1 + Q_1 P_1 (Q_2 P_2 - i)]\} \\ &= \frac{1}{4} \{P_1^2 Q_2^2 + P_2^2 Q_1^2 - (P_1 P_2 Q_2 Q_1 + Q_1 P_1 Q_2 P_2) + i[Q_1, P_1]\} \end{aligned} \tag{122}$$

while

$$J_2^2 = \frac{1}{4} \{P_1^2 P_2^2 + Q_1^2 Q_2^2 + (P_1 P_2 Q_1 Q_2 + Q_1 Q_2 P_1 P_2)\} \tag{123}$$

so that

$$J_1^2 + J_2^2 = H(P_1, Q_1)H(P_2, Q_2) - \frac{1}{4} \tag{124}$$

Then

$$\begin{aligned}J^2 &= \frac{1}{4} [H(P_1, Q_1) - H(P_2, Q_2)]^2 + H(P_1, Q_1)H(P_2, Q_2) - \frac{1}{4} \\ &= \frac{1}{4} [H(P_1, Q_1) + H(P_2, Q_2)]^2 - \frac{1}{4}.\end{aligned} \tag{125}$$

In the number representation $H(P_a, Q_a) \rightarrow N_a + \frac{1}{2}$, with $N_1 + N_2 = N$, it is

$$J^2 = \frac{1}{4} (N_1 + N_2 + 1)^2 - \frac{1}{4} = \frac{1}{2} N \left(\frac{1}{2} N + 1 \right) \tag{126}$$

This shows that the representation generated by J_1, J_2, J_3 contains all integer and semi-integer spins without degeneration, so it is a Schwinger representation [65]. Finally, the basis $\{|j, m\rangle\}$, $-j \leq m \leq j$, which diagonalizes $\{J^2, J_3\}$

is $\{|n_1 + n_2, n_1 - n_2\rangle = |n_+, n_-\rangle\}$, $-n_+ \leq n_- \leq n_+$ and is obtained with operators $N_{\pm} = N_1 \pm N_2$ of the isotropic bidimensional harmonic oscillator, by rotating the old basis $\{|n_1, n_2\rangle = |n_1\rangle |n_2\rangle\}$ (see, e.g., [66]).

The whole Hilbert space \mathcal{H} decomposes in a direct sum of finite dimensional subspaces \mathcal{H}^j of dimension $2j + 1$, which are the carrier spaces of the spin tomography. In each \mathcal{H}^j , the $(2j + 1)$ -dimensional spin tomography uses a set of fiducial operators, the eigenstates of the spin projection on z -axis J_3 , a generator of an irreducible unitary $(2j + 1)$ -dimensional representation τ of $SU(2)$:

$$\{A_0(j, m)\}_{m \in \sigma} = \{|jm\rangle \langle jm|\}_{m \in \sigma}, \quad \sigma = \{-j, \dots, j\} \quad (127)$$

The unitary transformations may be either the operators of the unitary group $U(2j + 1)$ or simply those of the representation τ of $SU(2)$, depending only on the angle parameters which determine a point on the sphere S^2 :

$$U(g), \quad g \in U(2j + 1); \quad \text{or} \quad U(g) = \tau(g), \quad g \in SU(2). \quad (128)$$

In the last case the tomogram describing a spin state is a probability distribution depending on random discrete spin projection along the direction determined by the angle parameters. General reconstruction formulae for the spin tomography were obtained in [27].

4 Quantum Mechanics in the tomographic picture

4.1 Superposition rule

Having put the tomographic approach within a general setting of states \mathcal{S} and observables \mathcal{O} , we may now ask how one can add probabilities to describe the superposition of pure quantum states which allows for the description of interference phenomena. For the case of two (orthonormalized) state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, given the probabilities (non-negative numbers) p_1 and p_2 , with $p_1 + p_2 = 1$, the superposition rule states that the linear combination

$$|\psi\rangle = \sqrt{p_1}|\psi_1\rangle + e^{i\varphi}\sqrt{p_2}|\psi_2\rangle \quad (129)$$

is also a (pure) state vector. Here the factor $e^{i\varphi}$ corresponds to the relative phase of the two state vectors. In the tomographic picture the formulation of the superposition principle should be discussed in terms of density states. In fact, since density states depend nonlinearly on state vectors:

$$\rho_1 = |\psi_1\rangle\langle\psi_1|, \quad \rho_2 = |\psi_2\rangle\langle\psi_2|, \quad (130)$$

a naive way to add them, that is $p_1\rho_1 + p_2\rho_2$, yields an operator corresponding to a mixed quantum state, different from a rank-one projector $\rho = |\psi\rangle\langle\psi|$ corresponding to the pure state of Eq. (129). However, a way to add rank-one projectors to obtain a rank one projector has been proposed in [67] and is given by

$$\rho = p_1\rho_1 + p_2\rho_2 + \sqrt{p_1p_2} \frac{\rho_1 P_0 \rho_2 + \rho_2 P_0 \rho_1}{\sqrt{\text{Tr}(\rho_1 P_0 \rho_2 P_0)}}, \quad (131)$$

where P_0 is a fiducial rank-one projector corresponding to the arbitrary phase factor $e^{i\varphi}$ of Eq. (129). The previous formula can be viewed as a purification of the mixed quantum state $p_1\rho_1 + p_2\rho_2$. If ρ_1 and ρ_2 are not orthogonal, the formula yields a non-normalized ρ , so it requires a normalization factor $(\text{Tr}\rho)^{-1}$. For the denominator to be different from zero we need to superpose only states which are not orthogonal to P_0 .

The composition rule for tomographic probabilities corresponding to the superposition of density states (Eq. (131)) reads, using $\vec{x} = (X, \mu, \nu)$

$$\mathcal{T}_\rho(\vec{x}) = p_1\mathcal{T}_{\rho_1}(\vec{x}) + p_2\mathcal{T}_{\rho_2}(\vec{x}) + \frac{\sqrt{p_1p_2}}{\sqrt{\text{Tr}(\rho_1 P_0 \rho_2 P_0)}} (\mathcal{T}_{\rho_1 P_0 \rho_2}(\vec{x}) + \mathcal{T}_{\rho_2 P_0 \rho_1}(\vec{x})). \quad (132)$$

This equations contains tomograms such as $\mathcal{T}_{\rho_1 P_0 \rho_2}(\vec{x})$ which is possible to express in terms of tomograms of ρ_1 and ρ_2 , by means of the star-product formalism.

In a star-product procedure, operators are replaced by their symbols, which are functions along the lines of section 3.2:

$$\hat{A} \longleftrightarrow f_A(\vec{x}). \quad (133)$$

The associative product of operators is mapped onto an associative product of symbols

$$\hat{A}\hat{B} \longleftrightarrow f_{AB}(\vec{x}) := f_A(\vec{x}) * f_B(\vec{x}) . \quad (134)$$

This star-product is described through a nonlocal integral kernel as

$$f_{AB}(\vec{x}) = \int f_A(\vec{x}_1) K(\vec{x}_1, \vec{x}_2, \vec{x}) f_B(\vec{x}_2) d\vec{x}_1 d\vec{x}_2 . \quad (135)$$

Tomographic probabilities (tomograms) can be considered as symbols of operators in a specific kind of star-product scheme.

In [28, 68] it is shown that the symplectic tomogram $\mathcal{T}_A(X, \mu, \nu)$ is the symbol of an operator A in a specific instance of star-product, with star-product kernel depending on the continuous variables $(X, \mu, \nu) = \vec{x}$. It is given by [69]:

$$K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = \frac{1}{4\pi^2} \delta(\mu(\nu_1 + \nu_2) - \nu(\mu_1 + \mu_2)) \quad (136)$$

$$\times \exp \left\{ \frac{i}{2} \left[\nu_1 \mu_2 - \nu_2 \mu_1 + 2X_1 + 2X_2 - \left(\frac{\nu_1 + \nu_2}{\nu} + \frac{\mu_1 + \mu_2}{\mu} \right) X \right] \right\} .$$

Thus, the addition rule for tomographic probabilities, corresponding to the superposition of density states Eq.(131), reads, using again $\vec{x} = (X, \mu, \nu)$

$$\begin{aligned} \mathcal{T}_\rho(\vec{x}) &= p_1 \mathcal{T}_{\rho_1}(\vec{x}) + p_2 \mathcal{T}_{\rho_2}(\vec{x}) \\ &+ \frac{\sqrt{p_1 p_2} \mathcal{T}_{\rho_1}(\vec{x}) * \mathcal{T}_0(\vec{x}) * \mathcal{T}_{\rho_2}(\vec{x}) + \mathcal{T}_{\rho_2}(\vec{x}) * \mathcal{T}_0(\vec{x}) * \mathcal{T}_{\rho_1}(\vec{x})}{\sqrt{C(\mathcal{T}_{\rho_1}(\vec{x}) * \mathcal{T}_0(\vec{x}) * \mathcal{T}_{\rho_2}(\vec{x}) * \mathcal{T}_0(\vec{x}))}} \end{aligned} \quad (137)$$

where, restoring (X, μ, ν) :

$$\text{Tr}(A) = C(\mathcal{T}_A(X, \mu, \nu)) = \int \mathcal{T}_A(X, \mu, \nu) e^{iX} \delta(\mu) \delta(\nu) dX d\mu d\nu . \quad (138)$$

We can describe the addition rule of Eq.(137) in the following terms. Given the tomograms \mathcal{T}_{ρ_1} and \mathcal{T}_{ρ_2} of the pure states ρ_1, ρ_2 , and the tomogram \mathcal{T}_0 of a fiducial chosen pure state P_0 , then the tomogram \mathcal{T}_ρ is the tomogram of the pure state $\rho = |\psi\rangle \langle\psi|$ of Eq.(131) corresponding to the linear superposition of Eq.(129). The formula Eq.(137) guarantees that if $\mathcal{T}_{\rho_1}, \mathcal{T}_{\rho_2}$ and \mathcal{T}_0 are tomographic probabilities of rank-one projectors, the result of the nonlinear summation \mathcal{T}_ρ is again a normalized tomographic probability corresponding to a superposition state that can be realized in nature.

The composition rule Eq.(137) has its partner in terms of Wigner functions of pure states. Let the Wigner function $\mathcal{W}_1(p, q)$ corresponding to a pure state $|\psi_1\rangle \langle\psi_1|$ be composed with the Wigner function $\mathcal{W}_2(p, q)$ corresponding to another pure state $|\psi_2\rangle \langle\psi_2|$ and let the composition $\mathcal{W}(p, q)$ correspond to the superposition vector $|\psi\rangle = \sqrt{p_1}|\psi_1\rangle + e^{i\varphi} \sqrt{p_2}|\psi_2\rangle$ with the parameters used in Eq.(137). Then one has [70], dropping the arguments (p, q) , the result:

$$\mathcal{W} = p_1 \mathcal{W}_1 + p_2 \mathcal{W}_2 + \sqrt{p_1 p_2} \frac{\mathcal{W}_1 * \mathcal{W}_0 * \mathcal{W}_2 + \mathcal{W}_2 * \mathcal{W}_0 * \mathcal{W}_1}{\sqrt{C(\mathcal{W}_1 * \mathcal{W}_0 * \mathcal{W}_2 * \mathcal{W}_0)}} \quad (139)$$

where, restoring the arguments (p, q) , for any operator A and its Weyl symbol $\mathcal{W}_A(p, q)$ one has

$$\text{Tr}(A) = C(\mathcal{W}_A(p, q)) = \frac{1}{2\pi} \int_A \mathcal{W}(p, q) dp dq . \quad (140)$$

The star product of two Weyl symbols is given by the formula

$$\begin{aligned} & \mathcal{W}_1(p, q) * \mathcal{W}_0(p, q) \\ &= \int \mathcal{W}_1(p_1, q_1) \mathcal{W}_0(p_2, q_2) K(p_1, q_1, p_2, q_2, p, q) dp_1 dq_1 dp_2 dq_2 \end{aligned} \quad (141)$$

with the Grönwald kernel

$$K(p_1, q_1, p_2, q_2, p, q) = \frac{1}{\pi^2} \exp [2i(qp_1 - q_1p + q_1p_2 - q_2p_1 + q_2p - p_2q)] .$$

One can easily see that the exponent in the r.h.s. of the above formula may be written as $4iS$, where S is the area of a triangle in the phase space with vertices in the points $(q_1, p_1), (q_2, p_2), (q, p)$.

4.2 Uncertainty relations

Having considered the superposition rule within the tomographic scheme we consider now the formulation of uncertainty relations. We present below the derivation of uncertainty relations in a general framework. Given an operator A , then it is obvious that

$$\langle A^\dagger A \rangle_\rho = \text{Tr}(\rho A^\dagger A) \geq 0 \quad (142)$$

for any density state ρ which is a non-negative hermitian operator with unity trace. Let A be, *for systems with one degree of freedom*, a linear combination of operators $\{B_k\}$ to be suitably chosen.

$$A = \sum_k c_k B_k . \quad (143)$$

Then

$$\langle A^\dagger A \rangle_\rho = \sum_{h,k} c_h^* c_k \langle B_h^\dagger B_k \rangle_\rho \geq 0 \quad (144)$$

that can be rewritten, dropping the label ρ and using commutator and anti-commutator, as

$$\sum_{h,k} c_h^* c_k \left\langle \frac{1}{2} [B_h^\dagger, B_k]_+ + \frac{1}{2} [B_h^\dagger, B_k] \right\rangle \geq 0. \quad (145)$$

Since c_h^*, c_k are arbitrary complex numbers, the positivity of the quadratic form means the positivity of the matrix

$$(S)_{hk} = \left\langle \frac{1}{2} [B_h^\dagger, B_k]_+ + \frac{1}{2} [B_h^\dagger, B_k] \right\rangle \quad (146)$$

The Sylvester criterion provides the necessary and sufficient condition for the positivity of the matrix in terms of its principal minors

$$M_1(S) \geq 0, \quad M_2(S) \geq 0, \quad \dots, \quad \det S \geq 0. \quad (147)$$

This general scheme allows to obtain uncertainty relations.

Thus, for position Q and momentum P we choose the error operators as B 's

$$B_1 = Q - \langle Q \rangle = \Delta Q, \quad B_2 = P - \langle P \rangle = \Delta P. \quad (148)$$

So, we get

$$S = \begin{pmatrix} \sigma_{QQ} & \sigma_{QP} \\ \sigma_{QP} & \sigma_{PP} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & \langle [Q, P] \rangle \\ -\langle [Q, P] \rangle & 0 \end{pmatrix} \quad (149)$$

where we have introduced the variances

$$\sigma_{QQ} = \langle Q^2 \rangle - \langle Q \rangle^2, \quad \sigma_{PP} = \langle P^2 \rangle - \langle P \rangle^2 \quad (150)$$

and the covariance

$$\sigma_{QP} = \frac{1}{2} \langle QP + PQ \rangle - \langle Q \rangle \langle P \rangle. \quad (151)$$

We then obtain from the positivity of the M_1 principal minors the obvious conditions

$$\sigma_{QQ} \geq 0, \quad \sigma_{PP} \geq 0, \quad (152)$$

and from the positivity of the determinant

$$\det S = \sigma_{QQ}\sigma_{PP} - \left(\sigma_{QP}^2 + \frac{1}{4} \langle -[Q, P] \rangle^2 \right) \geq 0 \quad (153)$$

the Schrödinger-Robertson uncertainty relation follows

$$\langle \Delta Q^2 \rangle \langle \Delta P^2 \rangle \geq \frac{1}{4} \langle [\Delta Q, \Delta P]_+^2 \rangle + \frac{1}{4}, \quad (154)$$

this reduces to the weaker Heisenberg uncertainty relation for uncorrelated states:

$$\left(\langle P^2 \rangle - \langle P \rangle^2 \right) \left(\langle Q^2 \rangle - \langle Q \rangle^2 \right) \geq \frac{1}{4}, \quad (\hbar = 1). \quad (155)$$

In view of Eq. (81), we can write Eq. (155) as:

$$\begin{aligned} & \left(\int X^2 \mathcal{T}(X, 0, 1) dX - \left[\int X \mathcal{T}(X, 0, 1) dX \right]^2 \right) \times \\ & \left(\int X^2 \mathcal{T}(X, 1, 0) dX - \left[\int X \mathcal{T}(X, 1, 0) dX \right]^2 \right) \geq \frac{1}{4} \end{aligned} \quad (156)$$

because $\mathcal{T}(X, 0, 1)$ is the momentum probability density and $\mathcal{T}(X, 1, 0)$ is the position probability density. The uncertainty relation in *covariant form* reads

$$\left(\int X^2 \mathcal{T}(X, \cos \theta, \sin \theta) dX - \left[\int X \mathcal{T}(X, \cos \theta, \sin \theta) dX \right]^2 \right) \times \quad (157)$$

$$\left(\int X^2 \mathcal{T}(X, -\sin \theta, \cos \theta) dX - \left[\int X \mathcal{T}(X, -\sin \theta, \cos \theta) dX \right]^2 \right) \geq \frac{1}{4} .$$

The obtained relation takes into account all the values of optical tomogram for all angles θ . It allows to use the values of experimentally measured optical tomogram to check the uncertainty relation of position and momentum by using a homodyne detector. This formula, that can be written in the short form $F(\theta) \geq 1/4$, provides a constraint on admissible quantum mechanical tomographic probabilities due to Heisenberg uncertainty relations. The experimental check of the Heisenberg uncertainty relations can be done using the positivity condition of the function (uncertainty function)

$$\Phi(\theta) := F(\theta) - \frac{1}{4} \geq 0 . \quad (158)$$

Such a checking can be done [71] (see also the discussion in [72]) using experimental data of [14, 15, 73, 74].

The tomographic expression for Schrödinger-Robertson relation requires the star product to represent the mean value of the anti-commutator.

For the case of angular momentum we choose selfadjoint B 's operators as

$$B_k = \Delta J_k \quad , \quad k = 1, 2, 3 \quad (159)$$

and get, recalling that $[J_h, J_k] = i\varepsilon_{hkl} J_l$:

$$S = \begin{pmatrix} \sigma_{J_1 J_1} & \sigma_{J_1 J_2} & \sigma_{J_1 J_3} \\ \sigma_{J_1 J_2} & \sigma_{J_2 J_2} & \sigma_{J_2 J_3} \\ \sigma_{J_1 J_3} & \sigma_{J_2 J_3} & \sigma_{J_3 J_3} \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & \langle J_3 \rangle & -\langle J_2 \rangle \\ -\langle J_3 \rangle & 0 & \langle J_1 \rangle \\ \langle J_2 \rangle & -\langle J_1 \rangle & 0 \end{pmatrix} \quad (160)$$

where again variances and covariances appear.

The three first principal minors give the obvious conditions of positivity of all the variances

$$\sigma_{J_1 J_1} \geq 0 , \quad \sigma_{J_2 J_2} \geq 0 , \quad \sigma_{J_3 J_3} \geq 0 . \quad (161)$$

The first of the three second principal minors yields a quadratic relation with indices 123

$$\sigma_{J_1 J_1} \sigma_{J_2 J_2} - \left(\sigma_{J_1 J_2}^2 + \frac{1}{4} \langle J_3 \rangle^2 \right) \geq 0 \quad (162)$$

while the others give quadratic relations obtained from the first by circular permutation of the indices 231 and 312.

Finally, the positivity of $\det S$ yields a cubic relation connecting the variances and covariances of all the components of the angular momentum, or spin projections. The tomographic expression for a star-product kernel providing such a cubic relation exists, even though cumbersome [75].

4.3 Classical and quantum distributions: examples

Some observations are now in order to discuss the conditions the uncertainty relations give for considering a ‘tomographic’ normalized function the quantum tomogram of a density state (the necessary and sufficient conditions for a function on phase space to be a Wigner function are discussed, e.g, in [76]). As we have observed previously, an exact condition stems out from the positivity of density states. Thus, given a ‘tomographic’ probability $\mathcal{T}(\vec{x})$ (either for spin variables or for continuous ones as position and momentum), the appropriate reconstruction formula can be used to get an operator $\varrho_{\mathcal{T}}$. Then, we can check whether its eigenvalues are non-negative. If they are, $\varrho_{\mathcal{T}}$ is a density state, *possibly non-normalized*, and \mathcal{T} is a quantum tomogram. However, this condition is rather formal and other, possibly more operative, conditions may be given. The ‘tomogram’ \mathcal{T} is assumed to satisfy the specific properties which hold for the tomographic probabilities in the given tomographic scheme, such as the homogeneity condition $\mathcal{T}(\lambda X, \lambda\mu, \lambda\nu) = \mathcal{T}(X, \mu, \nu)/|\lambda|$ in the symplectic case, for instance. Altogether these properties guarantee the hermiticity of $\varrho_{\mathcal{T}}$ and the normalization $\text{Tr}(\varrho_{\mathcal{T}}) = 1$, but they are not sufficient for the nonnegativity of $\varrho_{\mathcal{T}}$. For that, there are necessary, in general not sufficient, conditions like uncertainty relations. In fact, the nonnegativity of the operator $\varrho_{\mathcal{T}}$ guarantees the fulfilling of all available uncertainty relations because they stem from the equation $\langle A^\dagger A \rangle_{\varrho_{\mathcal{T}}} = \text{Tr}(A^\dagger A \varrho_{\mathcal{T}})/\text{Tr}(\varrho_{\mathcal{T}}) \geq 0$. Of course a complete (usually infinite) set of uncertainty inequalities is sufficient for the nonnegativity of $\varrho_{\mathcal{T}}$. However, such a completeness is a tautology, as in a sense it stands for ‘positivity’.

So, if the corresponding tomographic expressions of such inequalities are satisfied by \mathcal{T} , we have the necessary, or sufficient, conditions for \mathcal{T} to be a genuine quantum tomogram.

This method requires the use of the dual tomographic map [77] for the operator symbols $f_A^d(\vec{x}) := \langle A | G(\vec{x}) \rangle$, as for an observable A one has [78]:

$$\langle A \rangle_{\varrho_{\mathcal{T}}} = \text{Tr}(A \varrho_{\mathcal{T}}) = \langle A | \varrho_{\mathcal{T}} \rangle = \int \langle A | G(\vec{x}) \rangle \langle P(\vec{x}) | \varrho_{\mathcal{T}} \rangle d\vec{x} = \int \mathcal{T}(\vec{x}) f_A^d(\vec{x}) d\vec{x}$$

Thus, the Schrödinger-Robertson uncertainty relation can be written for the symplectic case as:

$$\begin{aligned} & \int f_{Q^2}^d(X, \mu, \nu) \mathcal{T}(X, \mu, \nu) dX d\mu d\nu \times \int f_{P^2}^d(X, \mu, \nu) \mathcal{T}(X, \mu, \nu) dX d\mu d\nu \\ & - \left(\int f_{\frac{1}{2}[Q, P]_+}^d(X, \mu, \nu) \mathcal{T}(X, \mu, \nu) dX d\mu d\nu \right)^2 \geq \frac{1}{4} \end{aligned} \quad (163)$$

where $\langle Q \rangle, \langle P \rangle$ are assumed to be zero, while the dual symbols are given by:

$$\begin{aligned} f_{Q^2}^d(X, \mu, \nu) &= \frac{1}{2\pi} \text{Tr}(Q^2 \exp[i(X - \mu Q - \nu P)]), \\ f_{P^2}^d(X, \mu, \nu) &= \frac{1}{2\pi} \text{Tr}(P^2 \exp[i(X - \mu Q - \nu P)]), \\ f_{\frac{1}{2}[Q, P]_+}^d(X, \mu, \nu) &= \frac{1}{2\pi} \text{Tr}\left(\frac{1}{2}[Q, P]_+ \exp[i(X - \mu Q - \nu P)]\right), \end{aligned} \quad (164)$$

and are known generalized functions [79, 80]. The Schrödinger-Robertson relation, written in terms of these generalized functions integrated with the \mathcal{T} function, provides a necessary condition for \mathcal{T} to be a quantum tomogram. We illustrate this analysis with the help of some examples.

Example 1 Let us consider the probability distribution

$$\frac{e^{-\frac{x^2}{\mu^2 + \nu^2}}}{\sqrt{\pi(\mu^2 + \nu^2)}}$$

and check whether it fulfills or not the uncertainty relations.

To this aim one has to evaluate dual symbols like

$$f_{Q^2}^d(X, \mu, \nu) = \frac{1}{2\pi} \text{Tr}(Q^2 e^{i(X - \mu Q - \nu P)}). \quad (165)$$

By using the Weyl symbols

$$A \rightarrow W_A(p, q) \quad (166)$$

the trace is readily evaluated as

$$\text{Tr}(AB) = \int \frac{W_A W_B}{2\pi} dp dq. \quad (167)$$

We have

$$\begin{aligned} W_{Q^2} &= q^2, \\ W_{P^2} &= p^2, \\ W_{\frac{1}{2}(PQ+QP)} &= pq, \\ W_{e^{i(X - \mu Q - \nu P)}} &= e^{i(X - \mu q - \nu p)}, \end{aligned} \quad (168)$$

so that

$$\begin{aligned} f_{Q^2}^d(X, \mu, \nu) &= \frac{1}{2\pi} \text{Tr}(Q^2 e^{i(X - \mu Q - \nu P)}) = \frac{1}{2\pi} \int \frac{q^2 e^{i(X - \mu q - \nu p)}}{2\pi} dp dq \\ &= e^{iX} \delta(\nu) \int \frac{q^2 e^{-i\mu q}}{2\pi} dq = -e^{iX} \delta(\nu) \delta''(\mu) \end{aligned} \quad (169)$$

and analogously

$$\begin{aligned} f_{P^2}^d(X, \mu, \nu) &= -e^{iX} \delta''(\nu) \delta(\mu), \\ f_{\frac{1}{2}(PQ+QP)}^d(X, \mu, \nu) &= -e^{iX} \delta'(\nu) \delta'(\mu). \end{aligned} \quad (170)$$

Thus we obtain

$$\begin{aligned}
\langle \Delta Q^2 \rangle &= - \int \frac{e^{-\frac{X^2}{\mu^2 + \nu^2}}}{\sqrt{2\pi(\mu^2 + \nu^2)}} e^{iX} \delta(\nu) \delta''(\mu) dX d\mu d\nu = \frac{1}{2}, \\
\langle \Delta P^2 \rangle &= - \int \frac{e^{-\frac{X^2}{\mu^2 + \nu^2}}}{\sqrt{2\pi(\mu^2 + \nu^2)}} e^{iX} \delta(\nu)'' \delta(\mu) dX d\mu d\nu = \frac{1}{2}, \\
\left\langle \frac{1}{2} [\Delta Q, \Delta P]_+ \right\rangle &= - \int \frac{e^{-\frac{X^2}{\mu^2 + \nu^2}}}{\sqrt{2\pi(\mu^2 + \nu^2)}} e^{iX} \delta'(\nu) \delta'(\mu) dX d\mu d\nu = 0.
\end{aligned} \tag{171}$$

so the Schrödinger-Robertson uncertainty relation

$$\langle \Delta Q^2 \rangle \langle \Delta P^2 \rangle \geq \frac{1}{4} \langle [\Delta Q, \Delta P]_+ \rangle^2 + \frac{1}{4} \tag{172}$$

is satisfied. In this case it is easy to reconstruct the quantum state ϱ_T , it is the ground state of a harmonic oscillator. This can be done or directly, by the use of Eq. (12) or in two steps, first by a Radon anti-transform of the tomogram yielding a Wigner function on phase space, from which the quantum state is readily obtained.

Example 2 An example of a non-quantum, classical tomographic probability distribution is provided by the following normalized distribution

$$\frac{1}{|\mu + \nu|} \chi_{I(0, \mu + \nu)}(X).$$

where $I(0, \mu + \nu)$ is the interval with extrema 0 and $\mu + \nu$. As it is homogeneous, it has the form of a symplectic tomographic distribution. We have

$$\begin{aligned}
\langle \Delta Q^2 \rangle &= - \int \frac{1}{|\mu + \nu|} \chi_{I(0, \mu + \nu)}(X) e^{iX} \delta(\nu) \delta''(\mu) dX d\mu d\nu \\
&= - \int \frac{1}{|\mu|} \chi_{I(0, \mu)}(X) e^{iX} \delta''(\mu) dX d\mu = - \int \frac{2 \sin \frac{\mu}{2}}{\mu} \exp \left[i \frac{\mu}{2} \right] \delta''(\mu) d\mu = \frac{4}{3}
\end{aligned} \tag{173}$$

Analogously

$$\langle \Delta P^2 \rangle = \frac{4}{3} \tag{174}$$

While

$$\begin{aligned}
\left\langle \frac{1}{2} [\Delta Q, \Delta P]_+ \right\rangle &= - \int \frac{1}{|\mu + \nu|} \chi_{I(0, \mu + \nu)}(X) e^{iX} \delta'(\nu) \delta'(\mu) dX d\mu d\nu \\
&= - \int \frac{2 \sin \frac{\mu + \nu}{2}}{\mu + \nu} \exp \left[i \frac{\mu + \nu}{2} \right] \delta'(\nu) \delta'(\mu) d\mu d\nu = \frac{4}{3}
\end{aligned} \tag{175}$$

so the uncertainty relations are violated:

$$\frac{4}{3} \cdot \frac{4}{3} \not\geq \left(\frac{4}{3} \right)^2 + \frac{1}{4}. \tag{176}$$

To obtain the classical distribution, we have to evaluate the Radon anti-transform of the tomographic distribution :

$$\begin{aligned} & \frac{1}{(2\pi)^2} \int \frac{1}{|\mu + \nu|} \chi_{I(0, \mu + \nu)}(X) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu \\ &= \frac{1}{(2\pi)^2} \int \frac{2 \sin(\frac{\mu + \nu}{2})}{\mu + \nu} \exp\left[i\left(\frac{\mu + \nu}{2} - \mu q - \nu p\right)\right] d\mu d\nu \end{aligned} \quad (177)$$

By introducing the new variables

$$\xi = \frac{\mu + \nu}{2}; \eta = \frac{\mu - \nu}{2} \Leftrightarrow \mu = \xi + \eta; \nu = \xi - \eta \quad (178)$$

we then get

$$\begin{aligned} & \frac{2}{(2\pi)^2} \int \frac{\sin \xi}{\xi} \exp[i\{\xi - \xi(q + p) - \eta(q - p)\}] d\xi d\eta \\ &= \frac{1}{\pi} \delta(q - p) \int \frac{\sin \xi}{\xi} \exp[i\{\xi - \xi(q + p)\}] d\xi \\ &= \chi_{[-1, 1]}(1 - (q + p)) \delta(q - p) = \chi_{[0, 1]}(q) \delta(q - p) \end{aligned} \quad (179)$$

Example 3 Another non-quantum, classical example is given by the positive homogeneous normalized distribution

$$\frac{1}{2|\mu + \nu|} \exp\left[-\frac{|X|}{|\mu + \nu|}\right]$$

Remembering that

$$\int \frac{1}{2|\mu + \nu|} \exp\left[-\frac{|X|}{|\mu + \nu|}\right] e^{iX} dX = \frac{1}{1 + |\mu + \nu|^2} \quad (180)$$

we get

$$\langle \Delta Q^2 \rangle = \langle \Delta P^2 \rangle = 2; \left\langle \frac{1}{2} [\Delta Q, \Delta P]_+ \right\rangle = 2 \quad (181)$$

so that again the uncertainty relations are violated, and the tomographic distribution is not quantum. With the change of variable

$$\xi = \mu + \nu; \eta = \mu - \nu \Leftrightarrow \mu = \frac{\xi + \eta}{2}; \nu = \frac{\xi - \eta}{2} \quad (182)$$

in the Radon anti-transform, it corresponds to the classical distribution:

$$\begin{aligned} & \frac{1}{(2\pi)^2} \int \frac{1}{1 + |\mu + \nu|^2} \exp[-i(\mu q + \nu p)] d\mu d\nu \\ &= \frac{1}{2(2\pi)^2} \int \frac{1}{1 + \xi^2} \exp\left[-i\frac{\xi}{2}(q + p) - i\frac{\eta}{2}(q - p)\right] d\xi d\eta \\ &= \frac{1}{2\pi} \delta\left(\frac{1}{2}(q - p)\right) \frac{\pi}{2} \exp\left[-\frac{|q + p|}{2}\right] = \frac{1}{2} \delta(q - p) \exp[-|q|] \end{aligned} \quad (183)$$

Example 4 The analogous approach to spin tomography is based on the necessary conditions of Eq.s (161, 162) and positivity of the determinant of the matrix S of Eq. (160). All the elements of S can be written in terms of spin tomograms and dual spin tomographic symbols of the observables $J_k, J_h J_k (k, h = 1, 2, 3)$, which can be obtained using the Gram-Schmidt operators for the spin tomographic star-product scheme given in [75]. For instance, consider in the qubit tomographic case the vector component

$$\mathcal{T}_{\alpha,\beta}(\Omega) = \alpha \cos^2 \frac{\theta}{2} + \beta \sin^2 \frac{\theta}{2} = \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(\alpha - \beta) \cos \theta, \quad (184)$$

the other component being $1 - \mathcal{T}_{\alpha,\beta}(\Omega)$, corresponding to the operator

$$\rho_{\alpha,\beta} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}. \quad (185)$$

The elements

$$\sigma_{J_h J_k} = \frac{1}{2} \langle J_h J_k + J_k J_h \rangle - \langle J_h \rangle \langle J_k \rangle \quad (186)$$

of the S matrix, in view of

$$J_h J_k + J_k J_h = \frac{1}{2} \delta_{hk} I_2, \quad (187)$$

read as

$$\int d\Omega \langle \delta_{hk} I_2 | G_\Omega \rangle \mathcal{T}_{\alpha,\beta}(\Omega) - \int d\Omega \langle J_h | G_\Omega \rangle \mathcal{T}_{\alpha,\beta}(\Omega) \int d\Omega \langle J_k | G_\Omega \rangle \mathcal{T}_{\alpha,\beta}(\Omega)$$

with

$$\int d\Omega \langle \delta_{hk} I_2 | G_\Omega \rangle \mathcal{T}_{\alpha,\beta}(\Omega) = \frac{1}{4}(\alpha + \beta) \delta_{hk}, \quad (188)$$

while

$$\langle J_1 | G_\Omega \rangle = \frac{3}{2\pi} \cos \phi \sin \theta \Rightarrow \langle J_1 \rangle = 0 \quad (189)$$

$$\langle J_2 | G_\Omega \rangle = \frac{3}{2\pi} \sin \phi \sin \theta \Rightarrow \langle J_2 \rangle = 0 \quad (190)$$

$$\langle J_3 | G_\Omega \rangle = \frac{3}{4\pi} \cos \theta \Rightarrow \langle J_3 \rangle = \frac{1}{2}(\alpha - \beta) \quad (191)$$

so that eventually

$$S = \frac{1}{4} \begin{bmatrix} (\alpha + \beta) & i(\alpha - \beta) & 0 \\ -i(\alpha - \beta) & (\alpha + \beta) & 0 \\ 0 & 0 & (\alpha + \beta) - (\alpha - \beta)^2 \end{bmatrix}. \quad (192)$$

Now S is non-negative iff $0 \leq \alpha, \beta \leq 1$. Then, $0 \leq \mathcal{T}_{\alpha,\beta}(\Omega) \leq 1$ is the component of a probability vector corresponding to the (non-normalized) density state $\rho_{\alpha,\beta}$. When the uncertainty relations are violated, $\mathcal{T}_{\alpha,\beta}(\Omega)$ is readily recognizable not to be a probability vector.

4.4 Equations of motion

For the tomographic description of quantum mechanics to be equivalent to other existing ones we have to consider the formulation of dynamical evolution, along with the integration of the equations of motion usually done by solving an eigenvalue problem. In the coming two subsections we are going to address these aspects.

4.4.1 Time evolution

The evolution of a state probability distribution in classical hamiltonian mechanics with potential $U(q)$ is given by the Liouville equation

$$\frac{\partial}{\partial t} f(q, p, t) + p \frac{\partial}{\partial q} f(q, p, t) - \frac{\partial}{\partial q} U(q) \frac{\partial}{\partial p} f(q, p, t) = 0 \quad (193)$$

We study now evolution in quantum mechanics using Wigner functions and tomograms. We recall the definition of Wigner function of a density matrix $\rho(q, q')$:

$$\mathcal{W}(p, q) = \int \rho(q + \frac{x}{2}, q - \frac{x}{2}) \exp(-ipx) dx . \quad (194)$$

The equations of motion for ρ , derived from the Schrödinger equations, give rise to equations of motion for $\mathcal{W}(p, q)$.

The evolution equation for the density matrix $\rho(q, q', t)$ is obtained from the Schrödinger equation for the wave function. In fact, for the Hamiltonian

$$H(P, Q) = \frac{1}{2} P^2 + U(Q); (m = 1) \quad (195)$$

one has in the position representation:

$$i \frac{\partial}{\partial t} \psi(q, t) = \left(-\frac{1}{2} \frac{\partial^2}{\partial q^2} + U(q) \right) \psi(q, t) \quad (196)$$

and

$$-i \frac{\partial}{\partial t} \psi^*(q', t) = \left(-\frac{1}{2} \frac{\partial^2}{\partial q'^2} + U(q') \right) \psi^*(q', t). \quad (197)$$

So, upon subtracting the above equations after multiplication by $\psi^*(q', t)$ and $\psi(q, t)$ respectively, we get the von Neumann evolution equation for the density matrix $\rho(q, q', t) = \psi(q, t) \psi^*(q', t)$ of the pure state ψ :

$$i \frac{\partial}{\partial t} \rho(q, q', t) = \left[-\frac{1}{2} \frac{\partial^2}{\partial q^2} + \frac{1}{2} \frac{\partial^2}{\partial q'^2} + U(q) - U(q') \right] \rho(q, q', t). \quad (198)$$

Equations of motion for Wigner function $\mathcal{W}(p, q)$ can be derived by using the following association among derivatives of the density matrix and the Wigner

function:

$$\begin{aligned}
\frac{\partial}{\partial q} \mathcal{W}(q, p) &\leftrightarrow \left(\frac{\partial}{\partial q} + \frac{\partial}{\partial q'} \right) \rho(q, q') \\
i \frac{\partial}{\partial p} \mathcal{W}(q, p) &\leftrightarrow (q - q') \rho(q, q') \\
q \mathcal{W}(q, p) &\leftrightarrow \frac{1}{2} (q + q') \rho(q, q') \\
ip \mathcal{W}(q, p) &\leftrightarrow \frac{1}{2} \left(\frac{\partial}{\partial q} - \frac{\partial}{\partial q'} \right) \rho(q, q')
\end{aligned} \tag{199}$$

and

$$\begin{aligned}
\frac{\partial}{\partial q} \rho(q, q') &\leftrightarrow \left(\frac{1}{2} \frac{\partial}{\partial q} + ip \right) \mathcal{W}(q, p) \\
\frac{\partial}{\partial q'} \rho(q, q') &\leftrightarrow \left(\frac{1}{2} \frac{\partial}{\partial q} - ip \right) \mathcal{W}(q, p) \\
q \rho(q, q') &\leftrightarrow \left(q + \frac{i}{2} \frac{\partial}{\partial p} \right) \mathcal{W}(q, p) \\
q' \rho(q, q') &\leftrightarrow \left(q - \frac{i}{2} \frac{\partial}{\partial p} \right) \mathcal{W}(q, p)
\end{aligned} \tag{200}$$

Then, using the previous correspondence rules, we finally obtain the evolution equation for the Wigner function $\mathcal{W}(q, p, t)$, which reads

$$\begin{aligned}
i \frac{\partial}{\partial t} \mathcal{W}(q, p, t) &= -\frac{1}{2} \left[\left(\frac{1}{2} \frac{\partial}{\partial q} + ip \right)^2 - \left(\frac{1}{2} \frac{\partial}{\partial q} - ip \right)^2 \right] \mathcal{W}(q, p, t) \\
&\quad + \left[U \left(q + \frac{i}{2} \frac{\partial}{\partial p} \right) - U \left(q - \frac{i}{2} \frac{\partial}{\partial p} \right) \right] \mathcal{W}(q, p, t) .
\end{aligned} \tag{201}$$

The Wigner function is real. The above equation can be written as

$$\left(\frac{\partial}{\partial t} + p \frac{\partial}{\partial q} \right) \mathcal{W}(q, p, t) = -i \left[U \left(q + \frac{i}{2} \frac{\partial}{\partial p} \right) - U \left(q - \frac{i}{2} \frac{\partial}{\partial p} \right) \right] \mathcal{W}(q, p, t) \tag{202}$$

These equations of motion may be called the Moyal evolution equations. The expansion of Eq. (202) up to first order in $(\frac{i}{2} \partial / \partial p)$ yields the classical Liouville equation (193)

A similar correspondence table allows us to derive the evolution equations for tomograms out of the evolution equation for the Wigner function. To get the evolution equation for the tomogram $\mathcal{T}_\rho(X, \mu, \nu, t)$ the correspondence table

which is obtained from the Radon transform formulae is

$$\begin{aligned}
\frac{\partial}{\partial q} \mathcal{W}(q, p) &\leftrightarrow \mu \frac{\partial}{\partial X} \mathcal{T}(X, \mu, \nu) \\
\frac{\partial}{\partial p} \mathcal{W}(q, p) &\leftrightarrow \nu \frac{\partial}{\partial X} \mathcal{T}(X, \mu, \nu) \\
q \mathcal{W}(q, p) &\leftrightarrow -\frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial X} \right)^{-1} \mathcal{T}(X, \mu, \nu) \\
p \mathcal{W}(q, p) &\leftrightarrow -\frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial X} \right)^{-1} \mathcal{T}(X, \mu, \nu)
\end{aligned} \tag{203}$$

Thus, from the Moyal equation we obtain

$$\left(\frac{\partial}{\partial t} - \mu \frac{\partial}{\partial \nu} \right) \mathcal{T}(X, \mu, \nu, t) = -i \left[U \left(-\frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial X} \right)^{-1} + \frac{i}{2} \nu \frac{\partial}{\partial X} \right) - c.c. \right] \mathcal{T}(X, \mu, \nu, t). \tag{204}$$

The "Planck constant expansion" of both Moyal and tomographic evolution equations is given by the potential energy expansion in powers of $(-\frac{i}{2}\partial/\partial p)$ for Moyal and $(-\frac{i}{2}\nu\partial/\partial X)$ for tomographic equation, respectively. For example, the Moyal equation for an harmonic oscillator is

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + p \frac{\partial}{\partial q} \right) \mathcal{W}(q, p, t) &= -\frac{i}{2} \left[\left(q + \frac{i}{2} \frac{\partial}{\partial p} \right)^2 - c.c. \right] \mathcal{W}(q, p, t) \\
&= q \frac{\partial}{\partial p} \mathcal{W}(q, p, t).
\end{aligned} \tag{205}$$

while the tomographic one reads

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \mu \frac{\partial}{\partial \nu} \right) \mathcal{T}(X, \mu, \nu, t) &= -\frac{i}{2} \left[\left(-\frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial X} \right)^{-1} + \frac{i}{2} \nu \frac{\partial}{\partial X} \right)^2 - c.c. \right] \mathcal{T}(X, \mu, \nu, t) \\
&= -\nu \frac{\partial}{\partial \mu} \mathcal{T}(X, \mu, \nu, t).
\end{aligned} \tag{206}$$

4.4.2 Eigenvalue problem

We may obtain the Moyal and the tomographic form of the eigenvalue equation if we start from the von Neumann stationary equation for the density matrix. In the position representation, we have

$$E \rho(q, q') = \frac{1}{2} \left[-\frac{1}{2} \frac{\partial^2}{\partial q^2} - \frac{1}{2} \frac{\partial^2}{\partial q'^2} + U(q) + U(q') \right] \rho(q, q'). \tag{207}$$

so that with the previous correspondence rules the Moyal equation for the Wigner function is

$$\begin{aligned} E\mathcal{W}(q, p) = & -\frac{1}{4} \left[\left(\frac{1}{2} \frac{\partial}{\partial q} + ip \right)^2 + \left(\frac{1}{2} \frac{\partial}{\partial q} - ip \right)^2 \right] \mathcal{W}(q, p) \\ & + \frac{1}{2} \left[U \left(q + \frac{i}{2} \frac{\partial}{\partial p} \right) + U \left(q - \frac{i}{2} \frac{\partial}{\partial p} \right) \right] \mathcal{W}(q, p). \end{aligned} \quad (208)$$

and the corresponding tomographic form of the equation:

$$\begin{aligned} E\mathcal{T}(X, \mu, \nu) = & -\frac{1}{4} \left[\left(\frac{1}{2} \mu \frac{\partial}{\partial X} - i \frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial X} \right)^{-1} \right)^2 + c.c. \right] \mathcal{T}(X, \mu, \nu) \\ & + \frac{1}{2} \left[U \left(-\frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial X} \right)^{-1} + \frac{i}{2} \nu \frac{\partial}{\partial X} \right) + c.c. \right] \mathcal{T}(X, \mu, \nu) \end{aligned} \quad (209)$$

for the energy levels E . In the usual pictures, the energy spectrum is obtained by solving an eigenvalue problem of the Hamiltonian operator in the carrier Hilbert space. Thus, for the n -th level, Eq. (203) is to give the result

$$\mathcal{T}_n(X, \mu, \nu) = \frac{e^{-\frac{X^2}{\mu^2 + \nu^2}}}{\sqrt{\pi(\mu^2 + \nu^2)}} \frac{1}{n! 2^n} H_n^2 \left(\frac{X}{\sqrt{\mu^2 + \nu^2}} \right). \quad (210)$$

However, neither the Wigner function $\mathcal{W}(q, p)$ nor the tomographic probability densities $\mathcal{T}(X, \mu, \nu)$ are vectors of a linear space. The eigenvalue problem in the Hilbert space of states is mapped onto the problem of solving integro-differential equations either of Moyal (von Neumann) form or equations for probability distributions. It is remarkable that in time evolution ruled by the tomographic equation, the function $\mathcal{T}(X, \mu, \nu, t)$ at any time t is positive and normalized if the initial tomogram is. This is obvious due to the correspondence chain rules substituted in the initial unitary evolution of von Neumann equation. Nevertheless it would be difficult to see it at a first glance directly in the tomographic evolution equation, without going back to the von Neumann equation.

4.5 Composite systems: separability and entanglement

To define the notion of separability of quantum state of a composite system let us discuss the example of two qubits. Each of the qubits has in its own Hilbert space of states the standard basis $\{|m\rangle\}$ where $m = \pm 1/2$ is the spin projection on the z -axis, that is

$$\left| +\frac{1}{2} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left| -\frac{1}{2} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (211)$$

The spin operator $\vec{S} = (S_x, S_y, S_z)$ is determined by the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ as

$$\vec{S} = \frac{1}{2}\vec{\sigma} \quad (\hbar = 1) \quad (212)$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (213)$$

Thus the basis $\{|m\rangle\}$ satisfies the eigenvalue problem

$$\sigma_z|m\rangle = m|m\rangle \quad . \quad (214)$$

For two qubits the Hilbert space of states is four-dimensional and it is obtained as tensor product of the two-dimensional Hilbert spaces of each qubit, so that we have four basis vectors

$$|m_1 m_2\rangle = |m_1\rangle|m_2\rangle \quad ; \quad m_1, m_2 = \pm\frac{1}{2} \quad (215)$$

The previous equation means

$$\left|+\frac{1}{2} + \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (216)$$

$$\left|+\frac{1}{2} - \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (217)$$

$$\left|-\frac{1}{2} + \frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (218)$$

$$\left|-\frac{1}{2} - \frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (219)$$

Any pure state of one qubit $|\psi\rangle$ can be expressed as superposition of the basis states

$$|\psi\rangle = \psi_1 \left|+\frac{1}{2}\right\rangle + \psi_2 \left|-\frac{1}{2}\right\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (220)$$

The density matrix for one qubit in case of a pure state reads

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \psi_1\psi_1^* & \psi_1\psi_2^* \\ \psi_2\psi_1^* & \psi_2\psi_2^* \end{pmatrix}. \quad (221)$$

The density matrix for a mixed state of one qubit

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad (222)$$

is such that

$$\rho = \rho^\dagger \quad ; \quad \text{Tr} \rho = 1 \quad (223)$$

and

$$\rho \geq 0 \iff \rho_{11} \geq 0 \quad ; \quad \rho_{22} \geq 0 \quad ; \quad \rho_{11}\rho_{22} - \rho_{12}\rho_{21} \geq 0 . \quad (224)$$

It can be presented in the form

$$\rho = \lambda_1 |\psi_1\rangle \langle \psi_1| + \lambda_2 |\psi_2\rangle \langle \psi_2| , \quad (225)$$

where $\lambda_1, \lambda_2 \geq 0$ are its (non-negative) eigenvalues and $|\psi_1\rangle, |\psi_2\rangle$ the corresponding eigenvectors.

The simply separable state of two qubits is defined as the factorized state

$$|\varphi\rangle = |\varphi_1\rangle |\varphi_2\rangle = \begin{pmatrix} \varphi_{11} \\ \varphi_{12} \end{pmatrix} \otimes \begin{pmatrix} \varphi_{21} \\ \varphi_{22} \end{pmatrix} = \begin{pmatrix} \varphi_{11}\varphi_{21} \\ \varphi_{11}\varphi_{22} \\ \varphi_{12}\varphi_{21} \\ \varphi_{12}\varphi_{22} \end{pmatrix} . \quad (226)$$

The density matrix of this state is the tensor product of the density states of each qubit:

$$\rho_\varphi = |\varphi\rangle \langle \varphi| = \rho_{\varphi_1} \otimes \rho_{\varphi_2} . \quad (227)$$

A separable state of two qubits is defined as a convex sum of the above factorized states

$$\rho(1, 2) = \sum_k \lambda_k \rho_{\varphi_1}^{(k)} \otimes \rho_{\varphi_2}^{(k)} . \quad (228)$$

In Eq.(228) the states $\rho_{\varphi_1}^{(k)}, \rho_{\varphi_2}^{(k)}$ satisfy the condition of purity

$$\left(\rho_{\varphi_{1,2}}^{(k)} \right) \left(\rho_{\varphi_{1,2}}^{(k)} \right) = \rho_{\varphi_{1,2}}^{(k)} . \quad (229)$$

However, this condition can be violated, as the general definition of separable two qubit state is: a state is separable iff its density matrix can be given the form

$$\rho(1, 2) = \sum_k \lambda_k \rho_1^{(k)} \otimes \rho_2^{(k)} \quad (230)$$

where $\sum_k \lambda_k = 1$ with $\lambda_k \geq 0$, while $\rho_1^{(k)}, \rho_2^{(k)}$ are density matrices of each qubit respectively.

The above separability condition is written for density states. A condition which holds for the corresponding tomograms may be obtained by calculating the tomograms of the two sides of the above equation. The tomogram of the state $\rho(1, 2)$ is the joint probability function:

$$\mathcal{T}_{\rho(1,2)}(m_1, m_2, U) = \langle m_1 m_2 | U \rho(1, 2) U^\dagger | m_1 m_2 \rangle . \quad (231)$$

As the tomogram of a simple separable state factorizes in the product of independent probabilities:

$$\begin{aligned} & \left\langle m_1 m_2 \left| U_1 \otimes U_2 (\rho_1 \otimes \rho_2) U_1^\dagger \otimes U_2^\dagger \right| m_1 m_2 \right\rangle \\ &= \left\langle m_1 \left| U_1 \rho_1 U_1^\dagger \right| m_1 \right\rangle \left\langle m_2 \left| U_2 \rho_2 U_2^\dagger \right| m_2 \right\rangle, \end{aligned} \quad (232)$$

we eventually get from Eq. (230) the tomographic separability condition in the form:

$$\mathcal{T}_{\rho(1,2)}(m_1, m_2, U_1 \otimes U_2) = \sum_k \lambda_k \mathcal{T}_{\rho_1}^{(k)}(m_1, U_1) \mathcal{T}_{\rho_2}^{(k)}(m_2, U_2). \quad (233)$$

The state is called entangled when the 4×4 -density matrix $\rho(1,2)$ cannot be presented in the form (230), or its tomogram $\mathcal{T}_{\rho(1,2)}$ in the form of Eq. (233).

Now we reformulate the introduced notions and definitions using the tomographic representations of the qubit states. This representation is characterized by a map from density matrices onto the family of the probability distributions which is invertible. We recall that for one qubit one has

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \longleftrightarrow \vec{\mathcal{T}}_\rho(\vec{n}) = \begin{pmatrix} \mathcal{T}_\rho(+\frac{1}{2}, \vec{n}) \\ \mathcal{T}_\rho(-\frac{1}{2}, \vec{n}) \end{pmatrix} \quad (234)$$

where the unit vector $\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ determines a point on the sphere S^2 . As it was observed in section 3.8, the tomogram here is presented in the form of a probability vector, whose components $\mathcal{T}_\rho(m, \vec{n}) = \text{Tr}(U^\dagger |m\rangle \langle m| U \rho)$ are the diagonal elements of the unitarily rotated density matrix:

$$\mathcal{T}_\rho(+\frac{1}{2}, \vec{n}) = (U \rho U^\dagger)_{11} \quad ; \quad \mathcal{T}_\rho(-\frac{1}{2}, \vec{n}) = (U \rho U^\dagger)_{22} \quad (235)$$

where

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} e^{i(\psi+\varphi)/2} & \sin \frac{\theta}{2} e^{i(\psi-\varphi)/2} \\ -\sin \frac{\theta}{2} e^{-i(\psi-\varphi)/2} & \cos \frac{\theta}{2} e^{-i(\psi+\varphi)/2} \end{pmatrix} \quad (236)$$

is a unitary matrix of the group $SU(2)$ parametrized by the usual Euler angles θ, ψ, φ . The matrix U transforms the spinors according to the rotation labeled by the Euler angles.

For the qubit the tomogram can be considered as a function on the group $SU(2)$ (in fact on the homogeneous space $SU(2)/U(1)$). The formulae can be presented in the form

$$\begin{aligned} \mathcal{T}_\rho(+\frac{1}{2}, U) &= |u_{11}|^2 \rho_{11} + |u_{12}|^2 \rho_{22} + (u_{12} u_{11}^* \rho_{21} + \text{c.c.}) \\ \mathcal{T}_\rho(-\frac{1}{2}, U) &= |u_{21}|^2 \rho_{11} + |u_{22}|^2 \rho_{22} + (u_{21} u_{22}^* \rho_{12} + \text{c.c.}). \end{aligned} \quad (237)$$

In terms of Euler angles the tomographic probability reads

$$\begin{aligned} \mathcal{T}_\rho(+\frac{1}{2}, U) &= \cos^2 \frac{\theta}{2} \rho_{11} + \sin^2 \frac{\theta}{2} \rho_{22} + \cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{-i\varphi} \rho_{21} + e^{i\varphi} \rho_{12}) \\ \mathcal{T}_\rho(-\frac{1}{2}, U) &= \sin^2 \frac{\theta}{2} \rho_{11} + \cos^2 \frac{\theta}{2} \rho_{22} - \cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{i\varphi} \rho_{12} + e^{-i\varphi} \rho_{21}) \end{aligned} \quad (238)$$

One can regard the qubit tomogram in the following manner. The density matrix $i\rho$ can be considered as a point in the Lie algebra $u(2)$ of the group $U(2)$; then the formula (235) defines an orbit of the unitary group in its Lie algebra.

Assume $\rho_{12} = \rho_{21} = 0$: the initial point of the orbit, i.e. $U = I$, is determined by the two non-negative numbers ρ_{11} and ρ_{22} which satisfy the simplex condition $\rho_{11} + \rho_{22} = 1$, that is a segment with extremes in $(1, 0), (0, 1)$ in \mathbb{R}^2 . Thus we may choose in the Lie algebra of the group $U(2)$ a subset, which is a simplex in an affine subspace modeled on the Cartan subalgebra of $SU(2)$. The formula for the tomogram (237) determines the orbit of the group $SU(2)$ in the simplex. Further details of this relation between qudit tomograms and points of a simplex are discussed in [81]. It is interesting to note that, under the action of the group, the orbit starting from any point of the simplex does not go out of the simplex. Thus one can define the tomographic probability as a map of the pairs $(\vec{\rho}, U)$ onto the points of the simplex $\tilde{T}_\rho(U)$. Here $\vec{\rho} = (\rho_{11}, \rho_{22})^T$ is a column vector and U is an element of the $SU(2)$ group. As we remarked in section 3.9, we could also use the full unitary group in Eq. (237).

The general case $\rho_{12} = \rho_{21}^* \neq 0$ is analogous due to the possibility of diagonalizing:

$$U_0^\dagger \rho U_0 = \begin{pmatrix} \tilde{\rho}_{11} & 0 \\ 0 & \tilde{\rho}_{22} \end{pmatrix}, \quad (239)$$

so the generic state ρ is again mapped onto the orbit of the group in the given simplex, but the initial point of the unitary transformation is shifted by the diagonalizing unitary matrix U_0 , i.e. now the element UU_0 moves the simplex point $(\tilde{\rho}_{11}, \tilde{\rho}_{22})$.

We stress that the initial points of the orbits belonging to the simplex can be considered as elements of a Cartan subalgebra of the unitary group belonging to non-negative weights (the Weyl chamber from which all other weights can be obtained by discrete reflections (Weyl group)). Thus the formula for the tomogram (237) can be written as:

$$\begin{aligned} \mathcal{T}_\rho(+\frac{1}{2}, U) &= |(UU_0)_{11}|^2 \tilde{\rho}_{11} + |(UU_0)_{12}|^2 \tilde{\rho}_{22} \\ \mathcal{T}_\rho(-\frac{1}{2}, U) &= |(UU_0)_{21}|^2 \tilde{\rho}_{11} + |(UU_0)_{22}|^2 \tilde{\rho}_{22}, \end{aligned} \quad (240)$$

in the form of a bi-stochastic map acting on the simplex where the shift matrix U_0 and simplex point coordinates $(\tilde{\rho}_{11}, \tilde{\rho}_{22})$ are connected with eigenvalues and eigenvectors of the density matrix ρ by Eq.(239) (for further details, see [81]). The equation (240) can be reinterpreted in the following way. The orthogonal 2×2 -matrices

$$M(U) = \begin{pmatrix} |u_{11}|^2 & |u_{12}|^2 \\ |u_{21}|^2 & |u_{22}|^2 \end{pmatrix} \quad (241)$$

belong to the semigroup of bistochastic 2×2 -matrices.

We recall that, in the n -dimensional case, if $e \in \mathbb{R}^n$ denotes the column vector with all components +1 and e^T its transpose, an $n \times n$ -matrix M is

called (column) stochastic iff $e^T M = e^T$ and bistochastic iff

$$Me = e \text{ and } e^T M = e^T. \quad (242)$$

All such M 's are non-negative matrices. For invertible matrices M , if we consider not only bistochastic matrices but also their inverse, which need not be non-negative any more, we get an open dense subset of $GL(n-1, \mathbb{R})$, the general linear group of $(n-1) \times (n-1)$ -matrices [81]. Then formula (240), when U varies on all the elements of the unitary group, provides an orbit of the semigroup of matrices $M(U)$ in the simplex. Thus the tomogram of the quantum qubit state is orbit of the semigroup in the simplex, which is a subset of the first positive chamber of the Cartan subalgebra of $SU(2)$ group. The statement is correct for unitary tomogram of any qudit state. Since a tomogram in the tomographic representation is identified with a quantum state, for a qudit one can say that the quantum state is the orbit of the semigroup of bistochastic maps $M(U)$, parametrized by the pairs $\{(U, \vec{T}_\rho(U))\}$ which are the graph of the tomogram, in the simplex which is a subset of a chosen Cartan subalgebra of $SU(n)$ group. Due to this picture we write the qudit analog of Eq.(240) in matrix form as

$$\vec{T}_\rho(U) = M(UU_0) \vec{\rho}. \quad (243)$$

Here $\vec{\rho} = (\tilde{\rho}_{11}, \tilde{\rho}_{22}, \dots, \tilde{\rho}_{nn})^T$ is the probability column vector consisting of the eigenvalues of ρ or, equivalently, a point in the simplex in the given subalgebra. The ortho-stochastic matrix $M(UU_0)$ has matrix elements

$$(M(UU_0))_{js} = |(UU_0)_{js}|^2. \quad (244)$$

The columns of the matrix U_0 are normalized eigenvectors of ρ , and a 'gauge' has been chosen by fixing the phase factors of U_0 and an ordering of both the components of $\vec{\rho}$ and the columns of U_0 so that $U_0^\dagger \rho U_0 = \text{diag} [\tilde{\rho}_{11}, \tilde{\rho}_{22}, \dots, \tilde{\rho}_{nn}]$.

The component of the vector $\vec{T}_\rho(U)$ are tomographic probabilities. They are defining a spin tomogram if one takes as $n \times n$ -matrix $U \in SU(n)$ the matrix of a $(2j+1 = n)$ -dimensional irreducible representation of the group $SU(2)$.

For two qubits the condition determining a separable state of a composite (bipartite) system can be rewritten in the form

$$\vec{T}_{\rho(1,2)}(U_1 \otimes U_2) = \sum_k \lambda_k \left(|U_1 U_{10}^{(k)}|^2 \otimes |U_2 U_{20}^{(k)}|^2 \right) \vec{\rho}_1^{(k)} \otimes \vec{\rho}_2^{(k)} \quad (245)$$

or

$$|(U_1 \otimes U_2) U_0(1,2)|^2 \vec{\rho}(1,2) = \sum_k \lambda_k \left(|U_1 U_{10}^{(k)}|^2 \otimes |U_2 U_{20}^{(k)}|^2 \right) \vec{\rho}_1^{(k)} \otimes \vec{\rho}_2^{(k)}, \quad (246)$$

where the notation $|A|^2$ means $(|A|^2)_{js} = |A_{js}|^2$.

Thus, given the (non-negative) eigenvalues and eigenvectors of the density matrix of two qubit systems $\rho(1,2)$, which are the components of the vector

$\vec{\rho}(1,2)$ and the corresponding columns of the unitary matrix $U_0(1,2)$ respectively, the state is separable if the vector in the left hand side of Eq.(246) is a convex sum of vectors, which may be written as tensor products of the tomographic probability vectors of each qubit, i.e., with eigenvalues given by the components of $\vec{\rho}_1, \vec{\rho}_2$ and eigenvectors given by the columns of the unitary matrices $U_{10}^{(k)}$ and $U_{20}^{(k)}$.

In other words, the state is separable if the semigroup orbit determined by semigroup $|(U_1 \otimes U_2)U_0(1,2)|^2$ acting on the simplex can be presented as a convex set of orbits of semigroups determined by semigroups $|U_1 U_{10}^{(k)}|^2$ and $|U_2 U_{20}^{(k)}|^2$ acting on their own simplex.

Now we illustrate the previous theory by discussing some examples.

Example 1 Let us consider the simplest example of simply separable state of two qubits: $|\uparrow\uparrow\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2$. The density matrix of this state is diagonal:

$$\rho(1,2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (247)$$

The eigenvectors of this matrix may be chosen as

$$\vec{u}_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \vec{u}_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \vec{u}_{03} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \vec{u}_{04} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \quad (248)$$

and the eigenvalues are $(1, 0, 0, 0)$, that is:

$$\vec{\rho}(1,2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (249)$$

The unitary matrix U_0 constructed from the eigenvectors is, of course, the identity matrix:

$$U_0(1,2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4. \quad (250)$$

The tomogram Eq.(246) of the state reads

$$\vec{T}_{\rho(1,2)}(U_1 \otimes U_2) = |U_1 \otimes U_2|^2 \vec{\rho}(1,2) \quad (251)$$

because

$$(U_1 \otimes U_2)U_0 = (U_1 \otimes U_2)I_4 = U_1 \otimes U_2 \quad (252)$$

and thus Eq.(251) yields

$$\vec{T}_{\rho(1,2)}(U_1 \otimes U_2) = \begin{pmatrix} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\ \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \end{pmatrix} = |U_1 U_{10}|^2 \vec{\rho}_1 \otimes |U_2 U_{20}|^2 \vec{\rho}_2 \quad (253)$$

Here: $U_{10} = U_{20} = I_2$; $\vec{\rho}_1 = \vec{\rho}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The formula (245) contains only one term, with $\lambda_k = 1$ times the tensor product of the 2-vectors

$$|U_1 U_{10}|^2 \vec{\rho}_1 = \begin{pmatrix} \cos^2 \frac{\theta_1}{2} \\ \sin^2 \frac{\theta_1}{2} \end{pmatrix} ; \quad |U_2 U_{20}|^2 \vec{\rho}_2 = \begin{pmatrix} \cos^2 \frac{\theta_2}{2} \\ \sin^2 \frac{\theta_2}{2} \end{pmatrix} \quad (254)$$

Example 2 Another example of simply separable state of two qubits is the state with both spin-projections oriented along the x -axis. The state vector reads

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (255)$$

Thus one has the density matrix of the two qubits state, which is a pure simple separable state:

$$\rho(1,2) = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad (256)$$

The eigenvalues of this matrix yield the vector:

$$\vec{\rho}(1,2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (257)$$

The corresponding eigenvectors of the matrix $\rho(1,2)$ may be fixed as

$$\vec{u}_{01} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} ; \vec{u}_{02} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} ; \vec{u}_{03} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} ; \vec{u}_{04} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} ; \quad (258)$$

The above vectors are the columns of the unitary matrix U_0 :

$$U_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (259)$$

The matrix $(U_1 \otimes U_2)U_0$ is a product:

$$(U_1 \otimes U_2)U_0 = \frac{1}{\sqrt{2}}U_1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}U_2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (260)$$

Here the matrices U_1 and U_2 have the form of Eq.(236).

The corresponding orthostochastic matrix is

$$|(U_1 \otimes U_2)U_0|^2 = \left| \frac{1}{\sqrt{2}}U_1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right|^2 \otimes \left| \frac{1}{\sqrt{2}}U_2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right|^2 \quad (261)$$

due to the obvious property

$$|a \otimes b|^2 = |a|^2 \otimes |b|^2. \quad (262)$$

From this orthostochastic matrix and the vector (257) we get the tomogram probability vector as:

$$\begin{aligned} \vec{T}_{\rho(1,2)}(U_1 \otimes U_2) &= |(U_1 \otimes U_2)U_0|^2 \vec{\rho}(1,2) \\ &= \frac{1}{4} \begin{pmatrix} |\cos \frac{\theta_1}{2} e^{i\varphi_1} + \sin \frac{\theta_1}{2} e^{-i\varphi_1}|^2 |\cos \frac{\theta_2}{2} e^{i\varphi_2} + \sin \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |\cos \frac{\theta_1}{2} e^{i\varphi_1} + \sin \frac{\theta_1}{2} e^{-i\varphi_1}|^2 |-\sin \frac{\theta_2}{2} e^{i\varphi_2} + \cos \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |-\sin \frac{\theta_1}{2} e^{i\varphi_1} + \cos \frac{\theta_1}{2} e^{-i\varphi_1}|^2 |\cos \frac{\theta_2}{2} e^{i\varphi_2} + \sin \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |-\sin \frac{\theta_1}{2} e^{i\varphi_1} + \cos \frac{\theta_1}{2} e^{-i\varphi_1}|^2 |-\sin \frac{\theta_2}{2} e^{i\varphi_2} + \cos \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \end{pmatrix}. \end{aligned} \quad (263)$$

The tomogram has the form of tensor product, corresponding to only one term with $\lambda_k = 1$ in Eq.(245):

$$\begin{aligned} \vec{T}_{\rho(1,2)}(U_1 \otimes U_2) &= \frac{1}{2} \begin{pmatrix} |\cos \frac{\theta_1}{2} e^{i\varphi_1} + \sin \frac{\theta_1}{2} e^{-i\varphi_1}|^2 \\ |-\sin \frac{\theta_1}{2} e^{i\varphi_1} + \cos \frac{\theta_1}{2} e^{-i\varphi_1}|^2 \end{pmatrix} \otimes \frac{1}{2} \begin{pmatrix} |\cos \frac{\theta_2}{2} e^{i\varphi_2} + \sin \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |-\sin \frac{\theta_2}{2} e^{i\varphi_2} + \cos \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \end{pmatrix} \end{aligned} \quad (264)$$

Finally, taking a convex sum of the density matrices (247) and (256):

$$\rho(1,2) = \cos^2 \delta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{4} \sin^2 \delta \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (265)$$

we get a separable state whose tomogram, by construction, has the form of the convex sum of Eq.(245), with $\lambda_1 = \cos^2 \delta$, $\lambda_2 = \sin^2 \delta$:

$$\begin{aligned} \vec{T}_{\rho(1,2)}(U_1 \otimes U_2) &= \cos^2 \delta \begin{pmatrix} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\ \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \\ \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \\ \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \end{pmatrix} + \\ &+ \frac{\sin^2 \delta}{4} \begin{pmatrix} |\cos \frac{\theta_1}{2} e^{i\varphi_1} + \sin \frac{\theta_1}{2} e^{-i\varphi_1}|^2 \cdot |\cos \frac{\theta_2}{2} e^{i\varphi_2} + \sin \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |\cos \frac{\theta_1}{2} e^{i\varphi_1} + \sin \frac{\theta_1}{2} e^{-i\varphi_1}|^2 \cdot |-\sin \frac{\theta_2}{2} e^{i\varphi_2} + \cos \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |-\sin \frac{\theta_1}{2} e^{i\varphi_1} + \cos \frac{\theta_1}{2} e^{-i\varphi_1}|^2 \cdot |\cos \frac{\theta_2}{2} e^{i\varphi_2} + \sin \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \\ |-\sin \frac{\theta_1}{2} e^{i\varphi_1} + \cos \frac{\theta_1}{2} e^{-i\varphi_1}|^2 \cdot |-\sin \frac{\theta_2}{2} e^{i\varphi_2} + \cos \frac{\theta_2}{2} e^{-i\varphi_2}|^2 \end{pmatrix}. \end{aligned} \quad (266)$$

Example 3 Let us now consider an example of an entangled two qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(\left| +\frac{1}{2} \right\rangle \left| -\frac{1}{2} \right\rangle + \left| -\frac{1}{2} \right\rangle \left| +\frac{1}{2} \right\rangle \right). \quad (267)$$

The density matrix reads

$$|\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (268)$$

the four eigenvalues yield the probability vector

$$\vec{\rho} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \quad (269)$$

The corresponding eigenvectors may be chosen as

$$\vec{u}_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{u}_{02} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \vec{u}_{03} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \vec{u}_{04} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (270)$$

and the unitary matrix U_0 diagonalizing the density matrix is

$$U_0 = [|\vec{u}_{01}, \vec{u}_{02}, \vec{u}_{03}, \vec{u}_{04}|] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (271)$$

The second column of the matrix UU_0 has the form:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} u_{12} + u_{13} \\ u_{22} + u_{23} \\ u_{32} + u_{33} \\ u_{42} + u_{43} \end{pmatrix} \quad (272)$$

Then the second column of the corresponding orthostochastic matrix $(M(UU_0))_{js} = |(UU_0)_{js}|^2$ is the vector

$$\vec{\rho} = \frac{1}{2} \begin{pmatrix} |u_{12} + u_{13}|^2 \\ |u_{22} + u_{23}|^2 \\ |u_{32} + u_{33}|^2 \\ |u_{42} + u_{43}|^2 \end{pmatrix}. \quad (273)$$

Thus, the tomogram of the entangled two qubit state (267) is the probability vector

$$\vec{T}_\rho(U) = M(UU_0) \vec{\rho} = \frac{1}{2} \begin{pmatrix} |u_{12} + u_{13}|^2 \\ |u_{22} + u_{23}|^2 \\ |u_{32} + u_{33}|^2 \\ |u_{42} + u_{43}|^2 \end{pmatrix}$$

For the subgroup $U = U_1 \otimes U_2$, where

$$U_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{121} & a_{22} \end{pmatrix} \quad ; \quad U_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad , \quad (274)$$

the tomogram of the state reads:

$$\vec{T}_\rho(U = U_1 \otimes U_2) = \frac{1}{2} \begin{pmatrix} |a_{11}b_{12} + a_{12}b_{11}|^2 \\ |a_{11}b_{22} + a_{12}b_{21}|^2 \\ |a_{21}b_{12} + a_{22}b_{11}|^2 \\ |a_{21}b_{22} + a_{22}b_{21}|^2 \end{pmatrix}. \quad (275)$$

By using the Euler angles as parameters:

$$U_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{121} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta_1}{2} e^{i(\psi_1 + \varphi_1)/2} & \sin \frac{\theta_1}{2} e^{i(\psi_1 - \varphi_1)/2} \\ -\sin \frac{\theta_1}{2} e^{-i(\psi_1 - \varphi_1)/2} & \cos \frac{\theta_1}{2} e^{-i(\psi_1 + \varphi_1)/2} \end{pmatrix}, \quad (276)$$

$$U_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{121} & b_{22} \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta_2}{2} e^{i(\psi_2 + \varphi_2)/2} & \sin \frac{\theta_2}{2} e^{i(\psi_2 - \varphi_2)/2} \\ -\sin \frac{\theta_2}{2} e^{-i(\psi_2 - \varphi_2)/2} & \cos \frac{\theta_2}{2} e^{-i(\psi_2 + \varphi_2)/2} \end{pmatrix}, \quad (277)$$

the components of the above tomogram are the following explicit functions of $(\vec{n}_1, \vec{n}_2) \in S^2 \otimes S^2$:

$$\mathcal{T}_\rho(+\frac{1}{2}, +\frac{1}{2}, \vec{n}_1, \vec{n}_2) = \frac{1}{2} [\cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \quad (278)$$

$$+ \cos \frac{\theta_1}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} (e^{i(\varphi_1 - \varphi_2)} + e^{-i(\varphi_1 - \varphi_2)})]$$

$$\mathcal{T}_\rho(+\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2) = \frac{1}{2} [\cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}$$

$$- \cos \frac{\theta_1}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} (e^{i(\varphi_1 - \varphi_2)} - e^{-i(\varphi_1 - \varphi_2)})]$$

$$\mathcal{T}_\rho(-\frac{1}{2}, +\frac{1}{2}, \vec{n}_1, \vec{n}_2) = \mathcal{T}_\rho(+\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2)$$

$$\mathcal{T}_\rho(-\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2) = \mathcal{T}_\rho(+\frac{1}{2}, +\frac{1}{2}, \vec{n}_1, \vec{n}_2)$$

The problem of separability amounts to the existence of a decomposition of the above probability vector as a convex sum, according to Eq.(245):

$$\begin{aligned} & \vec{T}_\rho(U_1 \otimes U_2) \\ &= \sum_k \lambda_k \left[\begin{pmatrix} \cos^2 \frac{\theta_k}{2} & \sin^2 \frac{\theta_k}{2} \\ \sin^2 \frac{\theta_k}{2} & \cos^2 \frac{\theta_k}{2} \end{pmatrix} \otimes \begin{pmatrix} \cos^2 \frac{\chi_k}{2} & \sin^2 \frac{\chi_k}{2} \\ \sin^2 \frac{\chi_k}{2} & \cos^2 \frac{\chi_k}{2} \end{pmatrix} \right] \tilde{\rho}_1^{(k)} \otimes \tilde{\rho}_2^{(k)} \\ &= \sum_k \lambda_k \begin{pmatrix} \cos^2 \frac{\theta_k}{2} \tilde{\rho}_{11}^{(k)} + \sin^2 \frac{\theta_k}{2} \tilde{\rho}_{12}^{(k)} \\ \sin^2 \frac{\theta_k}{2} \tilde{\rho}_{11}^{(k)} + \cos^2 \frac{\theta_k}{2} \tilde{\rho}_{12}^{(k)} \end{pmatrix} \otimes \begin{pmatrix} \cos^2 \frac{\chi_k}{2} \tilde{\rho}_{21}^{(k)} + \sin^2 \frac{\chi_k}{2} \tilde{\rho}_{22}^{(k)} \\ \sin^2 \frac{\chi_k}{2} \tilde{\rho}_{21}^{(k)} + \cos^2 \frac{\chi_k}{2} \tilde{\rho}_{22}^{(k)} \end{pmatrix} \end{aligned} \quad (279)$$

which eventually leads to the following equation

$$\begin{aligned} & \frac{1}{2} \left(\cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2) \right) \\ = & \sum_k \lambda_k \left(\cos^2 \frac{\theta_k}{2} \tilde{\rho}_{11}^{(k)} + \sin^2 \frac{\theta_k}{2} \tilde{\rho}_{12}^{(k)} \right) \left(\cos^2 \frac{\chi_k}{2} \tilde{\rho}_{21}^{(k)} + \sin^2 \frac{\chi_k}{2} \tilde{\rho}_{22}^{(k)} \right) \end{aligned} \quad (280)$$

with an infinite number of unknown variables $\lambda_k, \theta_k, \chi_k, \tilde{\rho}_{11}^{(k)}, \tilde{\rho}_{12}^{(k)}, \tilde{\rho}_{21}^{(k)}, \tilde{\rho}_{22}^{(k)}$. This equation has no solutions, so the decomposition of Eq.(279) is impossible for this tomographic probability vector: the state is entangled. The proof is given in the next subsection.

To resume, we have presented examples of tomograms for two pure simply separable states. One state corresponds to both spins directed along z -axes. The density matrix of this state in the natural basis has the form of Eq.(247) and the tomogram is given in the form of probability vector (253) with the tensor product form of probability vectors of Eq.(254). Another state corresponds to both spins directed along the x -axis. The density matrix of this state has the form (256) and the tomogram is given by (263). The tensor product form of the tomogram is given by Eq.(264). The separable, but not simply separable, mixed state with density matrix (265) has the tomogram given by Eq.(266) in the form of convex series (245), with only two nonzero terms in the series.

An example of entangled state is given by the pure state (267), with density matrix (268) and tomogram (279).

4.6 Inequalities

To prove that equation (280) has no solution let us use the following inequality valid for stochastic 4×4 -matrices which are a tensor product of two stochastic 2×2 -matrices, i.e.

$$M = M_1 \otimes M_2. \quad (281)$$

Let us introduce the matrix

$$I_0 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \quad (282)$$

Then we prove that

$$|\text{Tr}(MI_0)| \leq 2 \quad (283)$$

is a necessary condition for the validity of Eq.(281).

In fact, the stochastic 4×4 -matrix has the form:

$$M = \begin{pmatrix} p_1 M_2 & q_1 M_2 \\ p_2 M_2 & q_2 M_2 \end{pmatrix}, \quad (284)$$

where the stochastic matrices M_1 and M_2 read

$$M_1 = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}; M_2 = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}. \quad (285)$$

The columns of M_1 and M_2 are probability vectors: $p_1 + p_2 = q_1 + q_2 = s_1 + s_2 = t_1 + t_2 = 1$ and all the matrix elements are non-negative. The trace in Eq. (283) reads:

$$\begin{aligned} \text{Tr}(MI_0) &= p_1(s_1 - s_2) - p_2(s_1 - s_2) + p_1(t_1 - t_2) - p_2(t_1 - t_2) \\ &+ q_1(s_1 - s_2) - q_2(s_1 - s_2) - q_1(t_1 - t_2) + q_2(t_1 - t_2) \\ &= (p_1 - p_2)[(s_1 - s_2) + (t_1 - t_2)] + (q_1 - q_2)[(s_1 - s_2) - (t_1 - t_2)] \\ &=: p(s + t) + q(s - t) \end{aligned} \quad (286)$$

The differences p, q, s, t of the probabilistic distributions satisfy respectively the inequalities

$$|p_1 - p_2| \leq 1, |q_1 - q_2| \leq 1, |s_1 - s_2| \leq 1, |t_1 - t_2| \leq 1 \quad (287)$$

To prove that the modulus of the sum in Eq.(286) does not exceed the number 2, consider the function f :

$$f(p, q, s, t) := p(s + t) + q(s - t) \quad (288)$$

that is a harmonic function of the four variables p, q, s, t , which are constrained to belong to the hypercube $K_4 = \{|p| \leq 1, |q| \leq 1, |s| \leq 1, |t| \leq 1\}$. Then maximum and minimum values of f on K_4 are reached on the boundary of the hypercube. Note that f has the special properties that, when any number of its variables is taken constant, f is still harmonic in the remaining variables. Therefore f is harmonic when restricted on each face of the boundary of the hypercube, which is an hypercube K_3 of one less dimension. In each face K_3 the max and min of f lie on its boundary. By repeating this argument, eventually the max and min of f are found to lie on the vertices of the initial hypercube K_4 , which are $2^4 = 16$ points. For the given function it is then a trivial matter to check that $\max f = 2$ and $\min f = -2$.

The argument has an immediate generalization to any number of dimensions: **Proposition:** Let f be a function of n variables (x_1, \dots, x_n) harmonic on the hypercube $K_n = \{|x_i| \leq 1; i = 1, \dots, n\}$, such that f is still harmonic when restricted to any face $K_s, 0 < s \leq n - 1$, belonging to the boundary of K_n . Then f has its extrema on the 2^n vertices of K_n .

From inequality (283) it follows that

$$\left| \text{Tr} \left(I_0 \sum_k \lambda_k M_k \right) \right| \leq 2 \quad (289)$$

where $\lambda_k \geq 0, \sum_k \lambda_k = 1$ and the matrices M_k have the form of Eq.(281). Then, the proof of non-existence of solution to Eq.(280) is reduced to check the violation of inequality Eq.(289).

Let us construct the 4×4 -matrix M using the probability vector (279). We take this vector for four pairs of argument matrices, namely:

$$M = \|\vec{T}_\rho(U_1^{(1)} \otimes U_2^{(1)}), \vec{T}_\rho(U_1^{(1)} \otimes U_2^{(2)}), \vec{T}_\rho(U_1^{(2)} \otimes U_2^{(1)}), \vec{T}_\rho(U_1^{(2)} \otimes U_2^{(2)})\|. \quad (290)$$

Thus, the constructed matrix M is a function of eight angles: the angles θ_a, φ_a which are the Euler angles determining the 2×2 -matrix $U_1^{(1)}$, those θ_d, φ_d determining the 2×2 -matrix $U_1^{(2)}$, while θ_b, φ_b determine the 2×2 -matrix $U_2^{(1)}$ and θ_c, φ_c the 2×2 -matrix $U_2^{(2)}$.

If one assumes that the equality (279) is valid, then the constructed matrix (290) has the form of a convex sum $\sum_k \lambda_k M_k$ satisfying the inequality (289) for all values of the eight angles $\theta_a, \varphi_a, \theta_d, \varphi_d, \theta_b, \varphi_b, \theta_c, \varphi_c$. On the other hand we know the explicit form of such a matrix. In fact, the first column of this matrix is

$$\vec{T}_\rho(U_1^{(1)} \otimes U_2^{(1)}) = \begin{pmatrix} x_{ab} \\ \frac{1}{2} - x_{ab} \\ \frac{1}{2} - x_{ab} \\ x_{ab} \end{pmatrix} \quad (291)$$

where

$$x_{ab} = \frac{1}{2} \left(\cos^2 \frac{\theta_a}{2} \sin^2 \frac{\theta_b}{2} + \sin^2 \frac{\theta_a}{2} \cos^2 \frac{\theta_b}{2} + \sin \theta_a \sin \theta_b \cos(\varphi_a - \varphi_b) \right), \quad (292)$$

the second column is obtained from the first one (291) by the following replacement:

$$a \rightarrow a \quad , \quad b \rightarrow c \quad , \quad (293)$$

the third column by:

$$a \rightarrow d \quad , \quad b \rightarrow b \quad , \quad (294)$$

and the fourth column by:

$$a \rightarrow d \quad , \quad b \rightarrow c \quad . \quad (295)$$

Taking the trace of the obtained matrix M with the matrix I_0 of (282), we get the following function of the eight angles:

$$B = 4(x_{ab} + x_{ac} + x_{db} - x_{dc}) - 2 \quad (296)$$

Now we look for values of the angles $\theta_a, \varphi_a, \theta_d, \varphi_d, \theta_b, \varphi_b, \theta_c, \varphi_c$ for which $|B|$ exceeds 2 (in fact [82] the maximum of this function is equal to $2\sqrt{2}$). Such values do exist and this implies that the hypothesis (279) for the given probability vector is false. In particular, the maximum of $|B|$ is achieved when $x_{ab} = x_{ac} = x_{db} = 1 - x_{dc}$, and the corresponding stochastic matrix reads

$$M = \begin{pmatrix} \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} \\ \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} \\ \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} \\ \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2+\sqrt{2}}{8} & \frac{2-\sqrt{2}}{8} \end{pmatrix}. \quad (297)$$

It means that Eq. (280) has no solution. The inequality

$$|B| \leq 2 \tag{298}$$

is called Bell inequality [83] (or CHSH inequality [84]).

5 Conclusions

To conclude we point out the main aspects discussed in the paper. We reviewed the probability representation of quantum states, in which wave functions or density states are replaced by tomographic probability distributions containing a complete information on quantum states. The mathematical mechanism to construct all the possible probability descriptions of quantum states was clarified. It amounts to constructing complete sets of rank-one projectors in the Hilbert space of operators acting on the underlying Hilbert space of state vectors. These sets are complete or overcomplete. The tomograms depending on continuous variables were shown to be used both in the classical and quantum domains. The set of all tomographic functions describing the quantum states, the classical states or no states at all was characterized. The characterization is expressed in terms of inequalities, given by Eq.s (73), (74). We suggested a method of direct experimental checking the Heisenberg uncertainty relations [71]. In experiments [14, 15, 73, 74] the optical tomogram of a photon state $T(X, \theta)$ was measured in order to find the Wigner function $\mathcal{W}(q, p)$ by means of the Radon anti-transform of the tomogram. We pointed out that the new inequality (157) for the dispersion of the homodyne photon quadrature $X = \cos \theta Q + \sin \theta P$, expressed in terms of integrals containing the directly measured optical tomogram $T(X, \theta)$, provides the method of checking Heisenberg uncertainty relations. The accuracy of the experiment on checking the uncertainty relations is in one-to-one correspondence with measuring optical tomogram in the above experiments. The inequality (157) can be violated if one uses tomograms of classical states. The Heisenberg uncertainty relation checking can be possibly accompanied by a checking of the inequality for tomographic entropy discussed in [85].

The entanglement of spin system states was given in terms of properties of the spin tomograms. The Bell inequalities were shown to reflect the properties of entanglement in terms of properties of joint probability distributions (spin tomograms) to be expressed (or not expressed) as convex sums of joint probability distributions without correlations. The quantum spin states provide some bounds for the correlations. The Cirelson [82] bound $2\sqrt{2}$ corresponds to the Bell number characterizing a maximally entangled state of two qubits expressed in terms of the system tomograms properties. The tomographic-like joint probability distributions for which the Bell number is greater than that bound (the maximum can be equal to 4, see [86]) do not correspond neither to states with quantum spin correlations nor to classical correlations, characterized by a bound equal to 2. The stochastic matrix which has the Bell number 4 is, for example, of the form:

$$M = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{3}{4} \\ 0 & \frac{2}{3} & \frac{1}{2} & 0 \end{pmatrix}. \quad (299)$$

Suppose, in an experiment on measuring the spin state tomogram of two qubits, to get the probability vector (251) and the stochastic matrix, whose columns are the values of this vector for 4 pairs of Euler angles, to be just the above matrix

M. This result contradicts quantum mechanics. Moreover, the ‘classical’ states of two qubits have the bound 2. Thus, the value 4 of the Bell number corresponds to a tomogram which is neither quantum nor classical, and this is similar to the example with the scaled Wigner function for continuous variables discussed in section 3.5. We mean ‘classical’ in the following sense: in classical probability theory, given a distribution function of two random classical variables, there are two possibilities. One is that the random variables are uncorrelated, i.e, the joint probability distribution has a factorized form as a product of two probability distributions, describing statistical properties of each of the random variables. Another possibility is that the random variables are correlated, so the joint probability distribution is a convex sum of factorized probability distributions. The correlations described by such sums we call ‘classical’. In a sense, it is a terminology that refers to a class of situations in which the joint probability distribution of two correlated random variables is thought of as a convex mixture of distributions without correlations.

This picture for discrete spin variables is analogous to the picture with continuous variables, where tomograms violating both inequalities (73), (74) do not correspond neither to classical nor to quantum states.

We have shown that in tomographic approach to quantum mechanics an important role is played by semigroups and their orbits in simplexes, since from a geometrical point of view the tomograms, being probability distributions, are points of simplexes and their dependence on extra parameters provides some domains in the simplexes as orbits of semigroups. For qubit systems the semigroups are obtained from unitary matrices taking the map of their elements onto their square moduli. There exist generalizations of symplectic tomographic maps both for classical mechanics of a top [87] and quantum mechanics with using maps with curvilinear coordinate lines Radon transforms [88]. The analysis presented in this paper may be extended to those cases of generalized Radon transform. We hope to study these maps and their properties in future papers.

Among the variety of open problems, we should mention few of them which are more crucial for the full equivalence of the tomographic picture of quantum mechanics with the existing ones. We do not have yet a complete and autonomous characterization of tomograms whose inverse Radon transform gives rise to a Wigner function or to a classical probability distribution. Similarly we do not have a complete characterization of the continuity property of the Radon transform and its inverse. Some of those aspects play a relevant role to establish the topological properties of the star-product among the observable functions and the action of the observables on the states. We shall consider these interesting problems in a forthcoming paper.

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